Jan. 26, 2018
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Based on notes by Jon Tamir, Giulia Fanti and Frank Ong
Announcements

• Office Hours
  • Miki: Wednesdays 4-5pm, Cory 506
  • Nick: Mondays 11-12pm, Cory 504
  • Li-Hao: Wednesdays 2-3pm, Cory 557
• Lab 0 – due Monday Jan. 29
• HW 1 – due Wednesday Jan. 31
• Questions?
About today

• Properties of discrete-time systems
• Properties of LTI systems
• Review on linear regression
Discrete-time systems

1. **Memoryless**
   - \( y[n] \) depends only on \( x[n] \)

2. **Linear**
   - \( T\{a_1 x_1[n] + a_2 x_2[n]\} = a_1 T\{x_1[n]\} + a_2 T\{x_2[n]\} \)

3. **Time Invariant**
   - If \( y_1[n] = T\{x[n]\} \) and \( y_2[n] = T\{x[n - n_0]\} = y_1[n - n_0] \)

4. **Causal**
   - \( y[n] \) depends only on current and past values of \( x[n] \)

5. **BIBO Stable**
   - If \( |x[n]| \leq B_x \ \forall n \) then \( |y[n]| \leq B_y \ \forall n \)
What about LTI systems?

1. **Memoryless**
   - $y[n]$ depends only on $x[n]$

2. **Linear**
   - $T\{a_1 x_1[n] + a_2 x_2[n]\} = a_1 T\{x_1[n]\} + a_2 T\{x_2[n]\}$

   **Linear & Time-invariant system**

3. **Time Invariant**
   - If $y_1[n] = T\{x[n]\}$ and $y_2[n] = T\{x[n - n_0]\} = y_1[n - n_0]$

4. **Causal**
   - $y[n]$ depends only on current and past values of $x[n]$

5. **BIBO Stable**
   - If $|x[n]| \leq B_x \ \forall n$ then $|y[n]| \leq B_y \ \forall n$
Causality of LTI system

\[ y[n] = \sum_{k=-\infty}^{\infty} h[k] x[n-k] \]

Causal system: \( y \) should only depends on \( x \) with nonnegative delay \( k \)

Causal LTI system:

\[ h[n] = 0, \quad n < 0 \]
BIBO stability of LTI system

System output

\[ y[n] = \sum_{k=-\infty}^{\infty} h[k] x[n - k] \]

What’s the condition for \( h[n] \)

LTI system is BIBO
BIBO stability of LTI system

\[ y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k] \]

System output

\[ |y[n]| = \left| \sum_{k} h[k]x[n-k] \right| \]

\[ \leq \sum_{k} |h[k]|x[n-k] = \sum_{k} |h[k]| \cdot |x[n-k]| \]

\[ \leq B_x \sum_{k} |h[k]| \]

If \( \sum_{k} |h[k]| < \infty \)

LTI system is BIBO
BIBO stability of LTI system

System output

\[ y[n] = \sum_{k=-\infty}^{\infty} h[k] x[n - k] \]

If LTI system is BIBO

\[ \sum_{k} |h[k]| < \infty \]
BIBO stability of LTI system

System output

\[ y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k] \]

system BIBO \[ \Rightarrow \sum_{k} |h[k]| < \infty \]

\[ \sum_{k} |h[k]| = \infty \Rightarrow \text{system not BIBO} \]
**BIBO stability of LTI system**

System output
\[ y[n] = \sum_{k=-\infty}^{\infty} h[k] x[n - k] \]

Want to find \( x \) such that \( y \) is always not bounded!!

If LTI system is BIBO
\[ \sum_{k} |h[k]| < \infty \]

\[ y[0] = \sum_{k} h[k] x[-k] = \sum_{k} \frac{|h[k]|^2}{|h[k]|} = \sum_{k} |h[k]| = \infty \]
Let $T_1$ and $T_2$ be two separate systems and $T$ be the cascaded system:

\[ T(y[n]) = T(T_1(x[n])) \]

**True or False?**

- If $T_1$ is LTI and $T_2$ is not LTI, then $T$ cannot be LTI

**False**

Consider the system $T_1 = 0$. Then $T = 0$

- If $T_1$ is not LTI and $T_2$ is not LTI, then $T$ cannot be LTI

**False**

Consider the system $T_1 \{x\} = x^3$ and $T_2 \{x\} = \frac{1}{x^3}$. Then $T \{x\} = x$
A discrete-time system $H$ produces an output signal $y$ that is the symmetric part of the input:

$$y[n] = \frac{x[n] + x[-n]}{2}$$

Which of the following are true?

- The system must be LTI
- The system cannot be LTI
A discrete-time system $H$ produces an output signal $y$ that is the symmetric part of the input:

$$y[n] = \frac{x[n] + x[-n]}{2}$$

Which of the following are true?

- The system must be LTI
- **The system cannot be LTI**
Not time invariant:

- For $x_1[n] = \delta[n]$, then $y_1[n] = \delta[n]$
- For $x_2[n] = \delta[n - 1]$, then $y_2[n] = \frac{\delta[n-1] + \delta[n+1]}{2}$
- $y_1[0] = 1$ but $y_2[1] = \frac{1}{2}$

→ Not time invariant

(however, the system is linear)
For each of the following systems, determine if the system is (1) linear, (2) causal, (3) time-invariant, and (4) BIBO stable

Indicate Y for Yes, N for No, or X for cannot be determined

\[ y[n] = \cos(\sqrt{|n|})x[n] \]

- Linear? Y
- Causal? Y
- Time-invariant? N
- BIBO Stable? Y
For each of the following systems, determine if the system is (1) linear, (2) causal, (3) time-invariant, and (4) BIBO stable.

The response to an input of $\delta[n - 1]$ is $\left(\frac{1}{2}\right)^n u[n]$

- Linear? X
- Causal? X
- Time-invariant? X
- BIBO Stable? X
Solution 3b (from old exam)

The response to an input of $\delta[n - 1]$ is $\left(\frac{1}{2}\right)^n u[n]$

• System with impulse response: $h[n] = \left(\frac{1}{2}\right)^{n+1} u[n + 1]$
  $\rightarrow$ Linear. Not causal. time-invariant. Stable

• System that always outputs $\left(\frac{1}{2}\right)^n u[n]$, regardless of input

• System that outputs $\left(\frac{1}{2}\right)^n u[n]$ when the input is $\delta[n - 1]$ and $\infty$ otherwise.
Many signal processing problems can be formulated as a **least squares**, where we try to find model parameters that best fit the observed data. We will see this many, many times
Example: Linear regression. Suppose we observe five data points $x[k]$, where $k = \{-2, -1, 0, 1, 2\}$. We want to fit a line $x = mk + b$ by minimizing the squared distance between the line and the data points:
For each value of \( k \), we have a linear equation for our model:
Example, \( k = 2: x[2] = 2m + b \)

And we have a squared error with our data:
Example, \( k = 2: (x[2] - (b + 2m))^2 \)

Sum of squared errors: \( \sum_k (x[k] - (mk + b))^2 \)

→ In matrix form, \( \text{Error} = \frac{1}{2} \| x - K\beta \|_2^2 \) \quad \text{Error} = \frac{1}{2} \left\| \begin{pmatrix} x_{-2} \\ x_{-1} \\ x_0 \\ x_1 \\ x_2 \end{pmatrix} - \begin{bmatrix} -2 & 1 \\ -1 & 0 \\ 1 & 1 \\ 2 & 1 \end{bmatrix} \begin{pmatrix} m \\ b \end{pmatrix} \right\|_2^2 \)
Linear regression primer

To find the best fit from a least squares sense, minimize the sum of squared errors:

\[
\minimize_{m, b} \frac{1}{2} \left\| \begin{pmatrix} x_{-2} \\ x_{-1} \\ x_0 \\ x_1 \\ x_2 \end{pmatrix} - \begin{pmatrix} -2 & 1 \\ -1 & 1 \\ 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} m \\ b \end{pmatrix} \right\|^2_2 = \minimize_{\beta} \frac{1}{2} \|x - K\beta\|^2_2
\]
To solve for \(b\) and \(m\), take the derivative (gradient) with respect to \(b\) and to \(m\), and set to zero:

\[
\begin{align*}
\min_{\beta} & \quad \frac{1}{2} \| x - K\beta \|^2  \\
\text{subject to} & \quad K^T K\beta = K^T x = 0 \\
\end{align*}
\]

In Python,

```python
K = np.array([...])
x = np.array([...])
beta = np.linalg.solve(K, x)
```
Discrete-time Fourier transform

\[ X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} \quad \text{– Analysis} \]

\[ x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} \, d\omega \quad \text{– Synthesis} \]
What is the Discrete-time Fourier transform of the below signal $x[n]$?

hint: convolution of two signals, $x[n] = r[n] * r[n]$
What is the Discrete-time Fourier transform of the below signal $x[n]$?

hint: convolution of two signals, $x[n] = r[n] * r[n]$

Fourier space

$R(e^{j\omega}) \cdot R(e^{j\omega}) = X(e^{j\omega})$
Solution 4

\[ X(e^{j\omega}) = R(e^{j\omega})^2 = \left( \frac{\sin\left(\frac{3\omega}{2}\right)}{\sin\left(\frac{\omega}{2}\right)} \right)^2 \]

Normalized magnitude response: