EE123 Digital Signal Processing

Lecture 5C
The Discrete Wavelet Transform
Announcements

• Midterm I
  – Next Friday. 9am-11pm in class
  – Everything till today (including)
  – Open everything -- except electronics

• Pre-Lab II + Lab II Part I Due on Monday.
  – Install SDR drivers + Software on laptops
  – Look at different parts of the spectrum
  – Look at the effect of windowing on spectrum
From STFT to Wavelets

- Continuous time

\[
Sf(u, \Omega) = \int_{-\infty}^{\infty} f(t)w(t - u)e^{-j\Omega t} dt
\]

\[
Wf(u, s) = \int_{-\infty}^{\infty} f(t) \frac{1}{\sqrt{s}} \Psi^*(\frac{t - u}{s}) dt
\]

*Morlet - Grossmann

M. Lustig, EECS UC Berkeley
From STFT to Wavelets

\[ W_f(u, s) = \int_{-\infty}^{\infty} f(t) \frac{1}{\sqrt{s}} \Psi^*(\frac{t-u}{s}) dt \]

• The function \( \Psi \) is called a mother wavelet
  – Must satisfy:

\[ \int_{-\infty}^{\infty} |\Psi(t)|^2 dt = 1 \Rightarrow \text{unit norm} \]

\[ \int_{-\infty}^{\infty} \Psi(t) dt = 0 \Rightarrow \text{Band-Pass} \]
STFT and Wavelets “Atoms”

**STFT Atoms**
(with hamming window)

\[ w(t - u) e^{j\Omega t} \]

\[ \Omega_{hi} \]

\[ \Omega_{lo} \]

**Wavelet Atoms**

\[ \frac{1}{\sqrt{s}} \Psi \left( \frac{t - u}{s} \right) \]

\[ s = 1 \]

\[ s = 3 \]
Examples of Wavelets

- **Mexican Hat**
  \[ \Psi(t) = (1 - t^2)e^{-t^2/2} \]

- **Haar**
  \[ \Psi(t) = \begin{cases} 
  -1 & 0 \leq t < \frac{1}{2} \\
  1 & \frac{1}{2} \leq t < 1 \\
  0 & \text{otherwise} 
\end{cases} \]
Wavelets Transform

- Can be written as linear filtering

\[
W f(u, s) = \frac{1}{\sqrt{s}} \int_{-\infty}^{\infty} f(t) \Psi^* \left( \frac{t - u}{s} \right) dt
\]

\[
= \left\{ f(t) \ast \overline{\Psi_s}(t) \right\} (u)
\]

\[
\overline{\Psi_s} = \frac{1}{\sqrt{s}} \Psi \left( \frac{t}{s} \right)
\]

- Wavelet coefficients are a result of bandpass filtering
Example 2: “Bumpy” Signal

\[ \log(s) \]

SombreroWavelet

\[ u \]
Example 2: “Bumpy” Signal

\[ \text{log}(s) \]

SombreroWavelet

\[ u \]
Example 2: “Bumpy” Signal

log(s)

SombreroWavelet

M. Lustig, EECS UC Berkeley
Example 2: “Bumpy” Signal

\[ \log(s) \]

SombreroWavelet

\[ u \]
Wavelet Transform

• Many different constructions for different signals
  – Haar good for piece-wise constant signals
  – Battle-Lemarie’ : Spline polynomials

• Can construct Orthogonal wavelets
  – For example: dyadic Haar is orthonormal

\[ \overline{\Psi}_{i,n}(t) = \frac{1}{\sqrt{2^i}} \Psi \left( \frac{t - 2^i n}{2^i} \right) \]
\[ i \in \{0, 1, 2, \ldots\} \]
Orthonormal Haar - Basis functions

\[ \overline{\Psi}_{0,0}(t) = \frac{1}{\sqrt{2^0}} \Psi\left(\frac{t - 2^00}{2^0}\right) = \Psi(t) \]
Orthonormal Haar - Basis functions

\[ \overline{\Psi}_{0,0}(t) = \frac{1}{\sqrt{2^0}} \Psi\left( \frac{t - 2^0 0}{2^0} \right) = \Psi(t) \]

\[ \overline{\Psi}_{0,-1}(t) = \frac{1}{\sqrt{2^0}} \Psi\left( \frac{t + 2^0 1}{2^0} \right) = \Psi(t + 1) \]
Orthonormal Haar

\[ \Psi_{1,0}(t) = \frac{1}{\sqrt{2^1}} \psi\left(\frac{t + 2^1 0}{2^1}\right) = \frac{1}{\sqrt{2}} \psi\left(\frac{t}{2}\right) \]

Same scale
non-overlapping

Orthogonal
between scales
Orthonormal Haar

\[ \Psi_{1,0}(t) = \frac{1}{\sqrt{2}} \Psi \left( \frac{t - 2^1 0}{2^1} \right) = \frac{1}{\sqrt{2}} \Psi \left( \frac{t}{2} \right) \]

\[ \Psi_{1,-1}(t) = \frac{1}{\sqrt{2}} \Psi \left( \frac{t + 2^1 1}{2^1} \right) = \frac{1}{\sqrt{2}} \Psi \left( \frac{t + 2}{2} \right) \]
Orthonormal Haar

Same scale
non-overlapping

Orthogonal
between scales
Scaling function

\[ \Psi_{i,n}(t) = \frac{1}{\sqrt{2^i}} \Psi\left(\frac{t - 2^i n}{2^i}\right) \]

• Problem:
  – Every stretch only covers half remaining bandwidth
  – Need Infinite functions

recall, for chirp:
Scaling function

\[ \overline{\Psi}_{i,n}(t) = \frac{1}{\sqrt{2^i}} \psi\left(\frac{t - 2^n}{2^i}\right) \]

- **Problem:**
  - Every stretch only covers half remaining bandwidth
  - Need Infinite functions
- **Solution:**
  - Plug low-pass spectrum with a scaling function
Haar Scaling function

\[ \Psi(t) = \begin{cases} 
-1 & 0 \leq t < \frac{1}{2} \\
1 & \frac{1}{2} \leq t < 1 \\
0 & \text{otherwise}
\end{cases} \]

\[ \Phi(t) = \begin{cases} 
1 & 0 \leq t < 1 \\
0 & \text{otherwise}
\end{cases} \]
Back to Discrete

• Early 80’s, theoretical work by Morlett, Grossman and Meyer (math, geophysics)

• Late 80’s link to DSP by Daubechies and Mallat.

• From CWT to DWT not so trivial!

• Must take care to maintain properties
Discrete Wavelet Transform

\[ d_{s,u} = \sum_{n=0}^{N-1} x[n] \Psi_{s,u}[n] \]

\[ a_{s,u} = \sum_{n=0}^{N-1} x[n] \Phi_{s,u}[n] \]
Discrete Wavelet Transform

\[ d_{s,u} = \sum_{n=0}^{N-1} x[n] \Psi_{s,u}[n] \]

\[ a_{s,u} = \sum_{n=0}^{N-1} x[n] \Phi_{s,u}[n] \]
Example: Discrete Haar Wavelet

Haar for n=2

Equivalent to DFT₂!
Discrete Orthogonal Haar Wavelet

Haar for n=8

scaling \( \Phi_{20} \)

\( \Psi_{20} \)

\( \Psi_{10} \)

\( \Psi_{11} \)

\( \Psi_{00} \)

\( \Psi_{01} \)

\( \Psi_{02} \)

\( \Psi_{03} \)
Discrete Orthogonal Haar Wavelet

\[ |\mathcal{F}\{\Psi_{0x}(e^{j\omega})\}| \]

\[ \Psi_{00} \]

\[ \Psi_{01} \]

\[ \Psi_{02} \]

\[ \Psi_{03} \]
Discrete Orthogonal Haar Wavelet

$\Psi_{10}$

$\Psi_{11}$

$\mathcal{F}\{\Psi_{1x}(e^{j\omega})\}$

$\omega$

$t$
Discrete Orthogonal Haar Wavelet

scaling

\( \Phi_{20} \)

\( \Psi_{20} \)

\( \left| \mathcal{F}\{\Phi_{2x}(e^{j\omega})\} \right| \quad \left| \mathcal{F}\{\Psi_{2x}(e^{j\omega})\} \right| \)
Optional: stop decomposition at Level 1

scaling

$\Phi_{10}$

$\Phi_{11}$

$|\mathcal{F}\{\Phi_{1x}(e^{j\omega})\}|$