EE123
Digital Signal Processing
Lecture 2C
z-Transform
Today

• Last time:
  – DTFT - Ch 2

• Today:
  – finish DTFT
  – Z-Transform briefly!
  – Ch. 3
Frequency Response of LTI Systems

Check response to a pure frequency:

\[ y[n] = \sum_{k=-\infty}^{\infty} h[k] e^{j\omega_0(n-k)} \]

\[ = \left( \sum_{k=-\infty}^{\infty} h[k] e^{-j\omega_0 k} \right) e^{j\omega_0 n} \]

\[ H(e^{j\omega}) \big|_{\omega=\omega_0} \]
Frequency Response of LTI Systems

Check response to a pure frequency:

\[ e^{i\omega_n} \xrightarrow{\text{LTI}} y[n] \]

\[ H(e^{j\omega}) = \text{DTFT}\{h[n]\} \]

\[ y[n] = H(e^{j\omega}) \bigg|_{\omega=\omega_0} e^{j\omega_0 n} \]

Output is the same pure frequency, scaled and phase-shifted!

\[ e^{j\omega_0 n} \]

is an eigen function of LTI systems

Recall eigen vectors satisfy: \[ A\nu = \lambda\nu \]
Example 3

Frequency response of a causal moving average filter

\[ y[n] = \frac{x[n - M] + \cdots + x[n]}{M + 1} \]

Q: What type of filter is it?  A: Low-Pass

\[ h[n] = \frac{1}{M + 1} w[n - \frac{M}{2}] \]
Example 3 Cont.

Frequency response of a causal moving average filter

\[ h[n] = \frac{1}{M + 1} w[n - \frac{M}{2}] \]

Same as example 1, only: Shifted by N, divided by M+1, M=2N

\[ H(e^{j\omega}) = \frac{e^{-j\omega \frac{M}{2}}}{M + 1} \cdot \frac{\sin \left( (\frac{M}{2} + \frac{1}{2})\omega \right)}{\sin \left( \frac{\omega}{2} \right)} \]
Example 3 Cont.

Frequency response of a causal moving average filter

\[ H(e^{j\omega}) = e^{-j\omega \frac{M}{2}} \cdot \frac{\sin \left( (\frac{M}{2} + 1)\omega \right)}{M + 1} \cdot \frac{\sin \left( \frac{\omega}{2} \right)}{\sin \left( \frac{\omega}{2} \right)} \]

Not a sinc!
AT THE HOME OF THE
FOURIER TRANSFORM FAMILY...

I WISH SINCY
WOULD STOP
PLAYING WITH
THAT IMAGINARY
FRIEND OF HIS.

DON'T WORRY.
IT'S JUST A
PHASE HE'S
GOING THROUGH.
Example 4:

Impulse Response of an Ideal Low-Pass Filter

\[ h_{LP}[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_{LP}(e^{j\omega}) e^{j\omega n} d\omega \]

\[ = \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} e^{j\omega n} d\omega \]
Example 4

Impulse Response of an Ideal Low-Pass Filter

\[ h_{\text{LP}}[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_{\text{LP}}(e^{j\omega}) e^{j\omega n} d\omega \]

\[ = \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} e^{j\omega n} d\omega \]

\[ = \frac{1}{2\pi jn} e^{j\omega n} \bigg|_{-\omega_c}^{\omega_c} = 2j \sin(\omega_c n) \]

\[ = \frac{\sin(\omega_c n)}{\pi n} \]
Example 4

Impulse Response of an Ideal Low-Pass Filter

\[ h_{LP}[n] = \frac{\sin(\omega_c n)}{\pi n} \]

sampled “sinc”

Non causal! **Truncate** and **shift right** to make causal
Example 4

Impulse Response of an Ideal Low-Pass Filter

Non causal! **Truncate** and **shift right** to make causal

How does it change the frequency response?

Truncation:

\[ \tilde{h}_{LP}[n] = w_N[n] \cdot h_{LP}[n] \]

property 2.9.7:

\[
\tilde{H}_{LP}(e^{j\omega}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_{LP}(e^{j\theta}) W(e^{j(\omega-\theta)}) d\theta
\]

Periodic convolution
Example 4

We get “smearing” of the frequency response.
We get rippling.
The z-Transform

• Used for:
  – Analysis of LTI systems
  – Solving difference equations
  – Determining system stability
  – Finding frequency response of stable systems
Eigen Functions of LTI Systems

• Consider an LTI system with impulse response $h[n]$: 

$$x[n] \xrightarrow{LTI} Y[n]$$

• We already showed that $x[n] = e^{j\omega n}$ are eigen-functions.

• What if $x[n] = z^n = (re^{j\omega})^n$
Eigen Functions of LTI Systems

\[ y[n] = \sum_{k=-\infty}^{\infty} h[k] z^{n-k} \]

\[ = \left( \sum_{k=-\infty}^{\infty} h[k] z^{-k} \right) z^n = H(z) z^n \]

• \( x[n] = z^n \) are also eigen-functions of LTI Systems

• \( H(z) \) is called a transfer function \( H(e^{j\omega}) \)

• \( H(z) \) exists for larger class of \( h[n] \) than
The z Transform

\[ X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n} \]

• Since \( z = re^{j\omega} \)

\[ X(z) \big|_{z=e^{j\omega}} = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} = DTFT\{x[n]\} \]
Region of Convergence (ROC)

- The ROC is a set of values of $z$ for which the sum

\[ \sum_{n=-\infty}^{\infty} x[n] z^{-n} \]

Converges.
Region of Convergence (ROC)

- Example 1: Right-sided sequence \( x[n] = a^n u[n] \)

\[
X(z) = \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n
\]

Recall:

\[
1 + x + x^2 + \cdots = \frac{1}{1-x}, \text{ if } |x| < 1
\]

So:

\[
X(z) = \frac{1}{1 - az^{-1}}, \quad \text{ROC} = \{z : |z| > |a|\}
\]
Region of Convergence (ROC)

- Example 2: \( x[n] = \left( \frac{1}{2} \right)^n u[n] + \left( -\frac{1}{3} \right)^n u[n] \)

\[
X(z) = \frac{1}{1 - \frac{1}{2}z^{-1}} + \frac{1}{1 + \frac{1}{3}z^{-1}}
\]

\[
\text{ROC} = \{ z : |z| > \frac{1}{2} \} \cap \{ z : |z| > \frac{1}{3} \} = \{ z : |z| > \frac{1}{2} \} 
\]
Region of Convergence (ROC)

• Example 3: Left sided sequence \( x[n] = -a^n u[-n - 1] \)

\[
X(z) = \sum_{n=-\infty}^{-1} -a^n z^{-n} = \sum_{m=1}^{\infty} -a^{-m} z^m = 1 - \sum_{m=0}^{\infty} (a^{-1} z)^m
\]

if \(|a^{-1} z| < 1\), i.e., \(|z| < |a|\) then,

\[
X(z) = 1 - \frac{1}{1 - a^{-1} z}
\]

\[
= \frac{-a^{-1} z}{1 - a^{-1} z} = \frac{1}{1 - az^{-1}}
\]
Region of Convergence (ROC)

• Expression is the same as Example 1!
• $\text{ROC} = \{z: |z| < |a|\}$ is different

• The $z$-transform without ROC does not uniquely define a sequence!
Region of Convergence (ROC)

- Example 4: \( x[n] = -\left(\frac{1}{2}\right)^n u[-n - 1] + \left(-\frac{1}{3}\right)^n u[n] \)

\[
X(z) = \frac{1}{1 - \frac{1}{2}z^{-1}} + \frac{1}{1 + \frac{1}{3}z^{-1}}
\]

\[
\text{ROC} = \{ z : |z| < \frac{1}{2} \} \cap \{ z : |z| > \frac{1}{3} \}
\]

\[
= \{ z : \frac{1}{3} < |z| < \frac{1}{2} \}
\]

Same as example 2
Region of Convergence (ROC)

• Example 5: \( x[n] = \left( \frac{1}{2} \right)^n u[n] - \left( -\frac{1}{3} \right)^n u[-n - 1] \)

\[
\text{ROC} = \{ z : |z| > \frac{1}{2} \} \cap \{ z : |z| < \frac{1}{3} \} = 0
\]

• Example 6: \( x[n] = a^n, \) two sided \( a \neq 0 \)

\[
\text{ROC} = \{ z : |z| > a \} \cap \{ z : |z| < a \} = 0
\]
Region of Convergence (ROC)

- Example 7: Finite sequence \( x[n] = a^n u[n] u[-n + M - 1] \)

\[
X[z] = \sum_{n=0}^{M-1} a^n z^{-n} = \frac{1 - a^M z^{-M}}{1 - az^{-1}}
\]

\[
= \prod_{k=1}^{M-1} \left( 1 - ae^{j \frac{2\pi k}{M}} z^{-1} \right)
\]

\[
\text{ROC} = \{ z : |z| > 0 \}
\]
Region of Convergence (ROC)
Properties of ROC

- A ring or a disk in Z-plane, centered at the origin
- DTFT converges iff ROC includes the unit circle
- ROC can’t contain poles
Properties of ROC

- For finite duration sequences, ROC is the entire $z$-plane, except possibly $z=0$,

\[ X(z) = 1 + z^{-1} + z^{-2} \quad \text{ROC excludes } z = 0 \]

- \[ X(z) = 1 + z^{1} + z^{2} \quad \text{ROC excludes } z = \infty \]
Properties of the ROC

• For right-sided sequences: ROC extends outward from the outermost pole to infinity
  Examples 1, 2

• For left-sided: inwards from inner most pole to zero
  Example 3

• For two-sided, ROC is a ring - or do not exist
  Examples 4, 5, 6
Several Properties of the Z-transform

\[ x[n - n_d] \leftrightarrow z^{-n_d} X(z) \]

\[ z_0^n x[n] \leftrightarrow X\left(\frac{z}{z_0}\right) \]

\[ n x[n] \leftrightarrow -z \frac{dX(z)}{dz} \]

\[ x[-n] \leftrightarrow X(z^{-1}) \]

\[ x[n] \ast y[n] \leftrightarrow X(z)Y(z) \]

ROC at least ROC_x \cap ROC_y
Inversion of the z-Transform

• In general, by contour integration within the ROC

\[ x[n] = \frac{1}{2\pi j} \oint_C X(z)z^{n-1} \]

• Ways to avoid it:
  – Inspection (known transforms)
  – Properties of the z-transform
  – Power series expansion
  – Partial fraction expansion
  – Residue theorem

• Most useful is the inverse of rational polynomials

\[ X(z) = \frac{B(z)}{A(z)} \]  \hspace{1cm} \text{Why?}