EE123
Digital Signal Processing

Lecture 2B
D. T. Fourier Transform
Something Fun

• goTenna http://www.gotenna.com/#
  – Text messaging radio
  – Bluetooth phone interface
  – MURS VHF radio or 900MHz ISM
  – 2W (VHF), 1W (900MHz)
  – 0.5-5 mile range
  – encryption, mesh networking
  – 2 for 150$

• Lab 6 implements a similar approach -- (without the slick system integration)

• https://www.youtube.com/watch?v=Won_P_SN RHk
Discrete-Time LTI Systems

- The impulse response $h[n]$ completely characterizes an LTI system “DNA of LTI”

\[ y[n] = h[n] * x[n] \]

\[ y[n] = \sum_{N=-\infty}^{\infty} h[n - m] x[m] \]

Sum of weighted, delayed impulse responses!
BIBO Stability of LTI Systems

• An LTI system is BIBO stable iff $h[n]$ is absolutely summable

$$\sum_{k=-\infty}^{\infty} |h[k]| < \infty$$
BIBO Stability of LTI Systems

• Proof: “if”

\[ |y[n]| = \left| \sum_{k=-\infty}^{\infty} h[k] x[n - k] \right| \]

\[ \leq \sum_{k=-\infty}^{\infty} |h[k]| \cdot |x[n - k]| \leq B_x \]

\[ \leq B_x \sum_{k=-\infty}^{\infty} |h[k]| < \infty \]
BIBO Stability of LTI Systems

• Proof: “only if”

– suppose \( \sum_{k=-\infty}^{\infty} |h[k]| = \infty \)
  show that there exists bounded \( x[n] \) that gives unbounded \( y[n] \)

– Let: \( x[n] = \frac{h[-n]}{|h[-n]|} = \text{Sign}\{h[-n]\} \)

\[
y[n] = \sum h[k] x[n - k]
\]

\[
y[0] = \sum h[k] x[-k] = \sum h[k] h[k] / |h[k]| = \sum |h[k]| = \infty
\]
Discrete-Time Fourier Transform (DTFT)

\[ X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} x[k]e^{-j\omega k} \]

\[ x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n} d\omega \]

Why one is sum and the other integral?

Why use one over the other?

Alternative

\[ X(f) = \sum_{k=-\infty}^{\infty} x[k]e^{-j2\pi f k} \]

\[ x[n] = \int_{-0.5}^{0.5} X(f)e^{j2\pi fn} df \]
Example 1:

DTFT:

\[ W(e^{j\omega}) = \sum_{k=-N}^{N} e^{-j\omega k} \]

\[ = e^{-j\omega N} (1 + e^{j\omega} + \cdots + e^{j\omega 2N}) \]

Recall:

\[ 1 + p + p^2 + \cdots + p^M = \frac{1 - p^{M+1}}{1 - p} \]

\[ p = e^{j\omega} \]

\[ M = 2N \]
Example 1 cont.

DTFT:
Example 1 cont.

DTFT:

\[ W(e^{j\omega}) = e^{-j\omega N} \left( 1 + e^{j\omega} + \cdots + e^{j\omega 2N} \right) \]

\[ = e^{-j\omega N} \frac{1 - e^{j\omega (2N+1)}}{1 - e^{j\omega}} \]

\[ = e^{-j\omega N} \frac{e^{-j\omega N} - e^{j\omega N} e^{j\omega}}{1 - e^{j\omega}} \]

\[ = \frac{e^{-j\omega (N+\frac{1}{2})} - e^{j\omega (N+\frac{1}{2})}}{e^{-j\frac{\omega}{2}} - e^{j\frac{\omega}{2}}} \]

\[ = \frac{\sin[(N + \frac{1}{2})\omega]}{\sin(\frac{\omega}{2})} \]

periodic sinc
Example 1 cont.

\[ W(e^{j\omega}) = \frac{\sin[(N + \frac{1}{2})\omega]}{\sin\left(\frac{\omega}{2}\right)} \rightarrow (2N + 1) \text{ as } \omega \rightarrow 0 \]

also, \( \Sigma x[n] \)

= 1, why?
Properties of the DTFT

**Perodicity:**
\[ X(e^{j(\omega + 2\pi)}) = X(e^{j\omega}) \]

**Conjugate Symmetry:**
\[ X^*(e^{j\omega}) = X(e^{-j\omega}) \quad \text{if } x[n] \text{ is real} \]

\[
\begin{align*}
\Re \{ X(e^{-j\omega}) \} &= \Re \{ X(e^{j\omega}) \} \\
\Im \{ X(e^{-j\omega}) \} &= -\Im \{ X(e^{j\omega}) \}
\end{align*}
\]

Big deal for: MRI, Communications, more....
Half Fourier Imaging in MR

k-space (Raw Data) → Image

Complete based on conjugate symmetry
Half the Scan time!

Discrete Fourier transform
SSB Modulation

Real Baseband signal has conjugate symmetric spectrum

AM modulation (DSB-SC)

$m[n] \cos(\omega_0 n)$

Single sideband (USB) half bandwidth
Amateur radio on shortwaves often use SSB modulation

Example: Websdr

http://websdr.org

http://100.1.108.103:8902
Properties of the DTFT cont.

**Time-Reversal**

\[ x[n] \iff X(e^{i\omega}) \]

\[ x[-n] \iff X(e^{-i\omega}) \]

\[ = X^*(e^{i\omega}) \text{ if } x[n] \in \mathbb{Real} \]

If \( x[n] = x[-n] \) and \( x[n] \) is real, then:

\[ X(e^{j\omega}) = X^*(e^{j\omega}) \]

\[ \rightarrow X(e^{j\omega}) \in \mathbb{Real} \]
Q: Suppose:

\[ x[n] \iff X(e^{j\omega}) \]

? \iff \text{Re} \left\{ X(e^{j\omega}) \right\}

A: Decompose \( x[n] \) to even and odd functions

\[ x[n] = x_e[n] + x_o[n] \]

\[ x_e[n] := \frac{1}{2}(x[n] + x[-n]) \]

\[ x_o[n] := \frac{1}{2}(x[n] - x[-n]) \]

\[ x_e[n] + x_o[n] \rightarrow \text{Re} \left\{ X(e^{j\omega}) \right\} + j\text{Im} \left\{ X(e^{j\omega}) \right\} \]
Oops!
Properties of the DTFT cont.

Time-Freq Shifting/modulation:

\[
\begin{align*}
    x[n] & \iff X(e^{j\omega}) \\
    x[n - nd] & \iff e^{-j\omega nd} X(e^{j\omega}) \\
    e^{j\omega_0 n} x[n] & \iff X(e^{j(\omega - \omega_0)})
\end{align*}
\]

Good for MRI! Why
Example 2

What is the DTFT of:

\[ e^{j\pi n} \]

High Pass Filter

See 2.9 for more properties
Frequency Response of LTI Systems

Check response to a pure frequency:

\[ e^{j\omega n} \xrightarrow{\text{LTI}} y[n] \]

\[ y[n] = \sum_{k=-\infty}^{\infty} h[k] e^{j\omega_0 (n-k)} \]

\[ = \left( \sum_{k=-\infty}^{\infty} h[k] e^{-j\omega_0 k} \right) e^{j\omega_0 n} \]

\[ H \left( e^{j\omega} \right) \bigg|_{\omega = \omega_0} \]
Frequency Response of LTI Systems

Check response to a pure frequency:

\[ e^{j\omega n} \xrightarrow{\text{LTI}} y[n] \]

\[ H(e^{j\omega}) = \text{DTFT}\{h[n]\} \]

\[ y[n] = H(e^{j\omega}) \bigg|_{\omega=\omega_0} e^{j\omega_0 n} \]

Output is the same pure frequency, scaled and phase-shifted!

\[ e^{j\omega_0 n} \]

is an eigen function of LTI systems

Recall eigen vectors satisfy: \[ A\nu = \lambda\nu \]
Example 3

Frequency response of a causal moving average filter

\[ y[n] = \frac{x[n - M]}{M + 1} + \cdots + x[n] \]

Q: What type of filter is it?  A: Low-Pass

\[ h[n] = \frac{1}{M + 1} w[n - \frac{M}{2}] \]
Example 3 Cont.

Frequency response of a causal moving average filter

\[ h[n] = \frac{1}{M + 1} w[n - \frac{M}{2}] \]

Same as example 1, only: Shifted by N, divided by M+1, M=2N

\[ H(e^{j\omega}) = \frac{e^{-j\omega \frac{M}{2}}}{M + 1} \cdot \frac{\sin \left(\left(\frac{M}{2} + \frac{1}{2}\right)\omega\right)}{\sin\left(\frac{\omega}{2}\right)} \]
Example 3 Cont.

Frequency response of a causal moving average filter

\[ H(e^{j\omega}) = \frac{e^{-j\omega \frac{M}{2}}}{M + 1} \cdot \frac{\sin \left( \left( \frac{M}{2} + 1 \right)\omega \right)}{\sin \left( \frac{\omega}{2} \right)} \]

Not a sinc!

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Example 4:

Impulse Response of an Ideal Low-Pass Filter

\[ h_{LP}[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_{LP}(e^{j\omega}) e^{j\omega n} d\omega \]

\[ = \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} e^{j\omega n} d\omega \]
Example 4

Impulse Response of an Ideal Low-Pass Filter

\[ h_{LP}[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_{LP}(e^{j\omega})e^{j\omega n} d\omega \]

\[ = \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} e^{j\omega n} d\omega \]

\[ = \frac{1}{2\pi jn} \bigg|^{\omega_c}_{-\omega_c} e^{j\omega n} = 2j \sin(\omega_c n) \]

\[ = \frac{\sin(\omega_c n)}{\pi n} \]
Example 4

Impulse Response of an Ideal Low-Pass Filter

\[ h_{LP}[n] = \frac{\sin(\omega_c n)}{\pi n} \]

sampled “sinc”

Non causal! **Truncate** and **shift right** to make causal
Example 4

Impulse Response of an Ideal Low-Pass Filter

Non causal! Truncate and shift right to make causal

How does it changes the frequency response?

Truncation:

\[ \tilde{h}_{LP}[n] = w_N[n] \cdot h_{LP}[n] \]

property 2.9.7:

\[ \tilde{H}_{LP}(e^{j\omega}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_{LP}(e^{j\theta})W(e^{j(\omega-\theta)})d\theta \]

Periodic convolution
Example 4

We get “smearing” of the frequency response
We get rippling