An interesting issue arises when a windowed signal is sampled exactly at the edges. Referring to Figure 1, consider a rectangular window \( w(t) = \Pi(t/2) = u(t + 1) - u(t - 1) \), and sampling rate \( T_s = 1.0 \) sec. Let \( x(t) = 1 \), to consider effects of the window. (Note that in problem set 4, we did not sketch the sampled and windowed signal in time.)

Let the sampling function be \( p(t) = \sum_{n=-\infty}^{\infty} \delta(t - n) \).

Then

\[
\begin{align*}
x_\delta(t) &= w(t) * p(t) \\
&= \Pi(t/2)[\delta(t + 1) + \delta(t) + \delta(t - 1)] \\
&= \delta(t + 1)u(t + 1) + \delta(t) + \delta(t - 1)(1 - u(t - 1))] \\
&= 0.5\delta(t + 1) + \delta(t) + 0.5\delta(t - 1) \\
&= 0.5 \delta(t + 1) + \delta(t) + 0.5 \delta(t - 1)
\end{align*}
\]

if we take \( u(t = 0) = 0.5 \). We can show this is the case by calculating \( X_\delta(j\omega) \) and then using the inverse Fourier transform.

Calculate Fourier transforms:

\[
\begin{align*}
w(t) = \Pi(t/2) \rightarrow W(j\omega) &= \frac{2\sin \omega}{\omega} \\
p(t) = \sum_{n=-\infty}^{\infty} \delta(t - n) \rightarrow P(j\omega) = 2\pi \sum_{k = -\infty}^{\infty} \delta(\omega - k2\pi)
\end{align*}
\]

The sampled spectrum is obtained from convolution in frequency, with

\[
X_\delta(j\omega) = \frac{1}{2\pi} W(j\omega) * P(j\omega) = \frac{1}{2\pi} W(j\omega) * 2\pi \sum_{k = -\infty}^{\infty} \delta(\omega - k2\pi)
\]

The spectrum for the sampled window is then:

\[
X_\delta(j\omega) = \sum_{k=-\infty}^{\infty} W(j(\omega - 2\pi k))
\]

Several frequency points are easy to calculate: \( \omega = 0, 2\pi, 4\pi, ..., \omega = \pi, 3\pi, 5\pi, ..., \) and \( \omega = \frac{\pi}{2}, \frac{3\pi}{2}, ... \):

\[
\begin{align*}
X_\delta(j2\pi n) &= \sum_{k=-\infty}^{\infty} W(j2\pi(n - k)) = 2\delta[n - k] \\
X_\delta(j(2n + 1)\pi) &= \sum_{k=-\infty}^{\infty} W(j\pi(2n + 1 - 2k)) = \frac{2\sin \pi(2n + 2k + 1)}{\pi(2n + 2k + 1)} = 0
\end{align*}
\]

Figure 1: Block diagram of DFT processing steps.
Figure 2: Superposition of 3 sinc functions centered at $-2\pi, 0, 2\pi$.

We know that $X_\delta(j\omega)$ is periodic with period $2\pi$. Since sinc() is even,

$$X_\delta(\pm j\frac{\pi}{2}) = X_\delta(\pm j\frac{3\pi}{2}) = X_\delta(\pm j\frac{5\pi}{2}) = \ldots$$

Adding up all the ‘aliased” copies of the sincs at $\omega = j\frac{\pi}{2}$, we get:

$$X_\delta(j\frac{\pi}{2}) = \sum_{k=-\infty}^{\infty} W(j(2\pi k - \frac{\pi}{2})) = \sum_{k=-\infty}^{\infty} \frac{4\pi}{\pi(1 - 1/\pi)} = \sum_{n=0}^{\infty} -\frac{1^n}{2n+1}$$

For a single sinc:

$$W(j(2\pi k - \frac{\pi}{2})) = \frac{2\sin[2\pi k - \frac{\pi}{2}]}{2\pi k - \frac{\pi}{2}} = \frac{2\sin[2\pi k - \frac{\pi}{2}]}{\frac{\pi}{2}(4k-1)} = \frac{4}{\pi} \left[ -\frac{1}{4k-1} \right]$$

For $k = 0, 1, 2, 3, \ldots$ we get

$$\frac{4}{\pi} (1, \frac{-1}{3}, \frac{-1}{7}, \frac{-1}{11}, \ldots)$$

For $k = -1, -2, -3, \ldots$ we get

$$\frac{4}{\pi} (\frac{1}{5}, \frac{1}{9}, \frac{1}{13}, \ldots)$$

Using the samples at odd multiples of $\frac{\pi}{2}$ from eqn. (5), we get:

$$X_\delta(j\frac{\pi}{2}) = \sum_{k=-\infty}^{\infty} \frac{4\pi}{\pi(1 - 1/\pi)} = \sum_{n=0}^{\infty} -\frac{1^n}{2n+1}$$

Conveniently, eqn. (6) is just the Taylor series for $\tan^{-1}(1)$, i.e.:

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots$$

Thus $X_\delta(j\frac{\pi}{2}) = \frac{\pi^2}{4} = 1$

Perhaps surprisingly, it can be shown that

$$X_\delta(j\omega) = \cos(\omega) + 1.$$ 

This spectrum is obtained by the Fourier transform of $x_\delta(t)$:

$$x_\delta(t) = 0.5\delta(t + 1) + \delta(t) + 0.5\delta(t - 1) \rightarrow X_\delta(j\omega) = 0.5e^{j\omega} + 1 + 0.5e^{-j\omega} = \cos(\omega) + 1$$

A sum of shifted sincs tends to a sinusoid as suggested by the spectrum shown in Fig. 2.