**Discussion 3: Graphical Convolution and CT Fourier Series**

**1. Graphical Convolution**

Consider Problem 3e from PS2.

Given input, \( x(t) = \delta(t) + \frac{1}{2}\delta(t - 1) + \frac{1}{4}\delta(t - 2) \), and impulse response, \( h(t) = e^{-t}u(t) \), find the output using the convolution integral,

\[
y(t) = \int_{-\infty}^{\infty} x(t - \tau)h(\tau)d\tau
\]

Using the convolution integral, we found that this convolution has the form:

\[
y(t) = e^{-t}u(t) + \frac{1}{2}e^{-(t-1)}u(t-1) + \frac{1}{4}e^{-(t-2)}u(t-2)
\]

The “flip-and-slide” method of convolution is like watching the input “slide into” the system. If we flip \( x(\tau) \) about \( \tau = 0 \) and slide it to the right, we can see that the first impulse “hits” \( h(\tau) \) at \( t = 0 \). At \( t = 1 \), we’ve slid the flipped version of \( x(\tau) \) by 1 unit, and the 2nd impulse “hits” \( h(\tau) \). At \( t = 2 \), we’ve slid the flipped version of \( x(\tau) \) by 2 units, and the 3rd impulse “hits” \( h(\tau) \).

Practice the following graphical convolutions:

(a) Let \( h(t) \) be a time-shifted delta function: \( h(t) = \delta(t - 3) \). Let \( x(t) \) be a “tent function”:

(b) Let \( x[n] \) be a ramp starting at zero \( (x[n] = n \cdot u[n]) \) and let \( h[n] \) be two delta functions:

\( h[n] = \delta[n] + \delta[n - 1] \)

Find \( x(t) * h(t) \) graphically.

Find \( x(t) * h(t) \) graphically.

Check your graphical result with the analytical result.
2. Continuous Time Fourier Series

The Fourier Series coefficients for a given periodic signal, \( x(t) \), are given by the analysis equation:

\[
a_k = \frac{1}{T} \int_{<T>} x(t) e^{-j k \frac{2\pi}{T} t} dt
\]

Where \( T \) is the fundamental period of \( x(t) \). The signal can be written using the Fourier coefficients in the synthesis equation:

\[
x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j k \frac{2\pi}{T} t} dt
\]

Prove the following properties of CT Fourier Series:

(a) Linearity: Given \( x(t) \leftrightarrow a_k \) and \( y(t) \leftrightarrow b_k \) both periodic with fundamental period \( T \), show \( z(t) = Ax(t) + By(t) \leftrightarrow c_k = Aa_k + Bb_k \)

(b) Time-shifting: Given \( x(t) \leftrightarrow a_k \) with fundamental period \( T \), \( y(t) = x(t - t_0) \), show \( y(t) \leftrightarrow b_k = e^{-j k \frac{2\pi}{T} t_0} a_k \)

(c) Time-scaling: Given \( x(t) \leftrightarrow a_k \) with fundamental period \( T \), \( y(t) = x(\alpha t) \), show \( y(t) \leftrightarrow b_k = a_k \)

(d) Consider a quarter-period cosine:

\[
x_1(t) = \begin{cases} \cos(\omega_0 t) & \frac{2\pi}{\omega_0} \left( n - \frac{1}{8} \right) < t < \frac{2\pi}{\omega_0} \left( n + \frac{1}{8} \right) \\ 0 & \text{otherwise} \end{cases}
\]

This looks like a “truncated” cosine. Find the Fourier Series coefficients of this signal.