Professor Fearing EECS120/Discussion 2 GSI: Alanna Tran

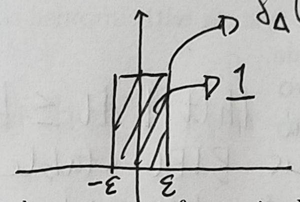
1. **Unit Impulse Function**

The unit impulse (Dirac delta) has the following properties:

let's call this $\delta_\Delta(t)$ where $\Delta = 2\varepsilon$

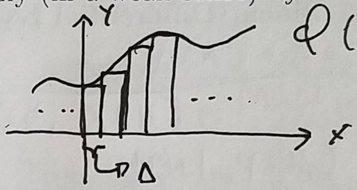
$$\delta(t) = \begin{cases} 0, t \neq 0 \\ \infty, t = 0 \end{cases}$$

$$\int_{-\epsilon}^{\epsilon} \delta(t)dt = 1$$



Remark 1 An ordinary function which is 0 everywhere except for a single point would have an integral value of 0 (in the Riemann integral sense). Thus, $\delta(t)$ cannot be defined like an ordinary function, but it can be defined mathematically (in a weak sense) by

$$\int_{-\infty}^{\infty} \phi(t)\delta(t)dt = \phi(0)$$



$\phi(t_\Delta)\,\delta_\Delta(t_\Delta)$

Delayed Delta

$$\int_{-\infty}^{\infty} \phi(t)\delta(t-t_0)dt = \phi(t_0)$$

to get the area:

$$\sum_{t_\Delta=\infty}^{\infty} \phi(t_\Delta)\underbrace{\delta(t-t_\Delta)\,\Delta}_{\text{area} = 1 \text{ of each block}}$$

similar to Riemann sum:

$$\lim_{\Delta \to 0} \sum_{-\infty}^{\infty} \phi(t_\Delta)\,\delta(t-t_\Delta)\,\Delta = \int_{-\infty}^{\infty} \phi(J)\,\delta(t-J)\,dJ$$

* *taken from Lillian Ratliff, Discussion 2 2014*

2. **Convolution**

Discrete Time

$$x[n] * h[n] = \Sigma_{k=-\infty}^{\infty} x[k]h[n-k]$$

Continuous Time

$$x(t) * h(t) = \int_{\tau=-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

Prove the following properties:

a. Identity

i. $x(t) * \delta(t)$

$$\int_{-\infty}^{\infty} x(\tau)\underbrace{\delta(t-\tau)}_{\text{only nonzero when } \tau = t.}d\tau = x(t)$$

only ~~have~~ nonzero when $\tau = t$.

ii. $x(t) * \delta(t-t_0)$

$$\int_{-\infty}^{\infty} x(\tau)\underbrace{\delta(t-t_0-\tau)}_{\text{only nonzero when } \tau = t - t_0}d\tau = x(t-t_0)$$

b. Commutative

$x(t) * h(t) = h(t) * x(t)$

$$x(t)*h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \longrightarrow \int_{\infty}^{-\infty} x(t-u)h(u)(-du)$$

$u = t - \tau$ $du = -d\tau$

$\tau = t - u$

$$= \int_{-\infty}^{\infty} x(t-u)h(u)du$$

$$= h(t) * x(t)$$

3. **B.I.B.O. Stability** *Bounded Input Bounded Output Stability*

For a system H, it is said to be B.I.B.O. stable if for any bounded input x the output y is bounded.

$$\|x(t)\| \le B < \infty$$

a. Given a system with impulse response $h(t) = (e^{-t} + e^{t})u(t)$, determine if the system is B.I.B.O. stable.

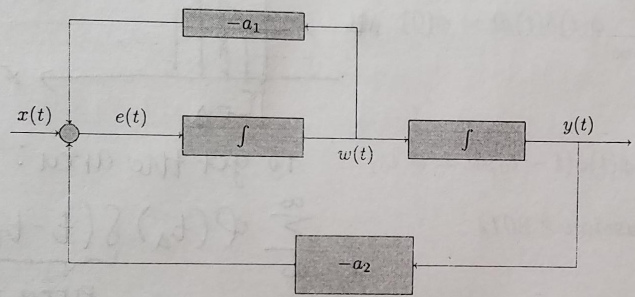
If $\int_{-\infty}^{\infty} |h(t)|\,dt \le H_B$ where H_B is a baund, then the system is BIBO stable.

$\int_{-\infty}^{\infty} |e^{-t} + e^{t} u(t)|\,dt = \int_{0}^{\infty} |e^{-t} + e^{t}|\,dt \to \infty$

↑ will grow to ∞

The system is not BIBO stable.

4. **Linear Differential Equations**



a. Write a differential equation that relates output ~~$y(t)$ and input $x(t)$~~

$\frac{dy(t)}{dt} = w(t)$ $\quad e(t) = \frac{dw(t)}{dt} = \boxed{\frac{d^2y(t)}{dt^2} = x(t) - a_1\frac{dy(t)}{dt} - a_2 y(t)}$

b. Consider the CT system whose input and output are related by:

$$\frac{dy(t)}{dt} + ay(t) = x(t)$$

where a is constant. Find $y(t)$ with the initial condition $y(0) = y_0$ and with input

$$x(t) = Ke^{-bt}u(t)$$

particular ↓ homogenons. ↓

guess: $y(t) = y_p(t) + y_h(t)$

$y_h(t) = Be^{-st}$

$\frac{dy_h(t)}{dt} + ay_h(t) = 0$

$-Bse^{-st} + aBe^{-st} = 0$

$(a-s) = 0$

$s = a.$

guess:

$y_p(t) = Ae^{-bt}$

$\frac{dy_p(t)}{dt} + ay_p(t) = x(t)$

$-Abe^{-bt} + aAe^{-bt} = Ke^{-bt}u(t)$

$A = \frac{K}{a-b}$ for $t>0$

b/c only for $t>0$ ↓

$y(t) = Be^{-at} + \frac{K}{a-b}e^{-bt}u(t)$

$y(0) = y_0 = B + \frac{K}{a-b}$

$B = y_0 - \frac{K}{a-b}.$

for $t>0$,

for $t<0$

$B = y_0$

$y(t) = y_0e^{-at} + \left(\frac{K}{a-b}e^{-bt} - \frac{K}{a-b}e^{-at}\right)u(t)$

time invariant: input $\rightarrow x[n-n_0]$

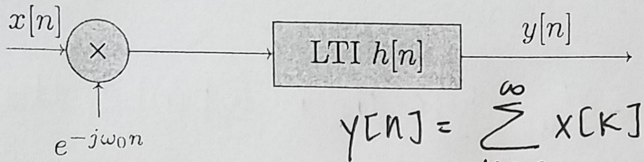
$$y[n] = \sum_{-\infty}^{\infty} x[k-n_0]\, e^{-jw_0 k}\, h[n-k]$$

$$y[n-n_0] = \sum_{-\infty}^{\infty} x[k-n_0]\, e^{-jw_0 (k-n_0)}\, h[n-k]$$

not equal, so not Time invariant.

5. LCCDE

Consider a system S with input $x[n]$ and output $y[n]$ related according to the block diagram in the figure below. The input $x[n]$ is multiplied by $e^{-j\omega_0 n}$ and the product is passed through a stable LTI system with impulse response $h[n]$.



$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\, e^{-jw_0 k}\, h[n-k]$$

a. Is the system S linear? Time invariant?

linear: input $\rightarrow \alpha x_1[n] + \beta x_2[n] \rightarrow y[n] = \sum_{-\infty}^{\infty} (\alpha x_1[k] + \beta x_2[k])\, e^{-jw_0 k}\, h[n-k]$

$$= \sum_{-\infty}^{\infty} \alpha x_1[k]\, e^{-jw_0 k}\, h[n-k] + \sum_{-\infty}^{\infty} \beta x_2[k]\, e^{-jw_0 k}\, h[n-k]$$

$$= \alpha y_1[n] + \beta x_2[n]$$

✓ is linear.

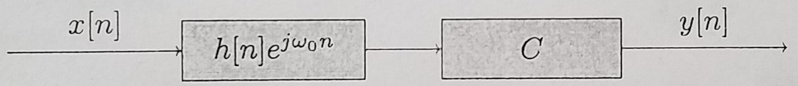
b. Is the system S stable?

$h[n] \rightarrow$ stable

$|e^{-jw_0 n}| \leq 1$

$\therefore$ if $x[n]$ is bounded, $x[n]e^{-jw_0 n}$ is bounded and the output $y[n]$ will be as well $\rightarrow$ yes.

c. Specify a system C such that the block diagram in the figure below represents an alternative way of expressing the input-output relationship of the system S.



$$y[n] = \sum_{-\infty}^{\infty} x[k]\, e^{-jw_0 k}\, h[n-k]$$

By commutative property of conv. & or a change of variables:

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]\, x[n-k]\, e^{-jw_0 (n-k)}$$

$$= \sum_{k=-\infty}^{\infty} h[k]\, x[n-k]\, e^{-jw_0 n}\, e^{jw_0 k}$$

$$= e^{-jw_0 n} \sum_{k=-\infty}^{\infty} h[k]\, e^{jw_0 k}\, x[n-k]$$

C is a system that multiplies the output of the first system by $e^{-jw_0 n}$.