Discussion 12: Minimum-Phase/All-Pass Decomposition and Steady-State Error

1. Minimum-Phase/All-Pass Decomposition

In many applications, we want to cancel the effect of a system on the system input, so we would like to apply a filter that is the exact inverse of the system in order to "undo" the distortion. However, systems with zeros outside the unit circle (in Z-transform) will result in inverted transfer functions that are not causal. By using the Min-phase/all-pass decomposition, we can find a filter that at least "undoes" the *magnitude* of the distortion, but leaves some phase distortion.

Exercise:

a) Consider the transfer function, $H_1(z) = \frac{(1+3z^{-1})}{\left(1+\frac{1}{2}z^{-1}\right)}$, which might represent the distortion on a transmission line. Draw the pole-zero diagram for $H_1(z)$.

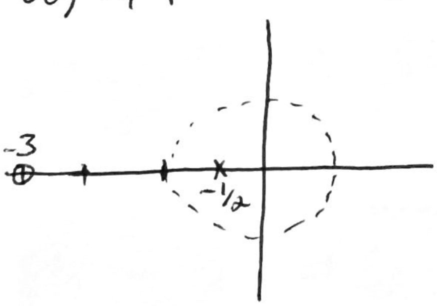
b) What is the "inverse" filter, or, in other words, what function $H_{inv}(z)$ exactly gives $H_{inv}(z)H_1(z) = 1$? Why can't we use this function, $H_{inv}(z)$, in a practical application?

c) Find some $H_{AP}(z)$ (all-pass filter) and $H_{MP}(z)$ ("min-phase" filter) such that $H_1(z) = H_{AP}(z)H_{MP}(z)$.

d) Using this "all-pass" decomposition, construct a new inverse filter such that $|H_{inv}(z)H_1(z)| = 1$.

Extra Practice: consider the transfer function, $H_2(z) = \frac{\left(1+\frac{3}{2}e^{+j\pi/4}z^{-1}\right)\left(1+\frac{3}{2}e^{-j\pi/4}z^{-1}\right)}{\left(1-\frac{1}{3}z^{-1}\right)}$. Find the all-pass/min-phase decomposition for $H_2(z)$.

a) H_1 pole-zero:

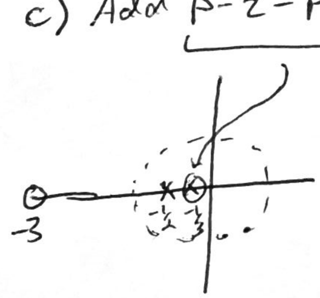


b) We would want $H_{inv}(z) = \frac{1}{H_1(z)}$

$H_{inv} = \frac{1+\frac{1}{2}z^{-1}}{1+3z^{-1}}$

This has pole outside unit circle, so H_{inv} isn't stable and causal. Not realizable

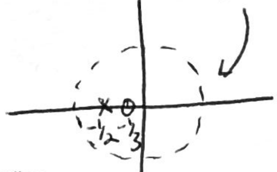
c) Add p-z-pair @ $-\frac{1}{3}$



Allpass filter

$$\boxed{H_{AP} = \frac{1}{3}\frac{(1+3z^{-1})}{\left(1+\frac{1}{3}z^{-1}\right)}}$$

Min-phase filter



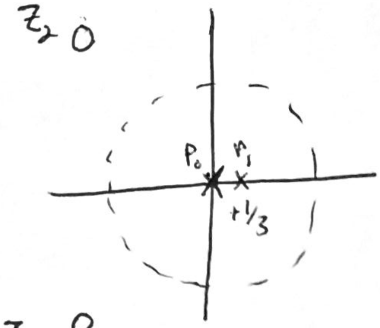
$$\boxed{H_{MP} = 3\,\frac{\left(1+\frac{1}{3}z^{-1}\right)}{\left(1+\frac{1}{2}z^{-1}\right)}}$$

d) we choose $\boxed{H_{inv} = \frac{1}{H_{MP}} = \frac{1}{3}\frac{\left(1+\frac{1}{3}z^{-1}\right)}{\left(1+\frac{1}{2}z^{-1}\right)}}$

H_{inv}, causal, stable

Extra practice:

$$H_2(z) = \frac{\left(1+\frac{3}{2}e^{+j\pi/4}z^{-1}\right)\left(1+\frac{3}{2}e^{-j\pi/4}z^{-1}\right)}{\left(1-\frac{1}{3}z^{-1}\right)}$$

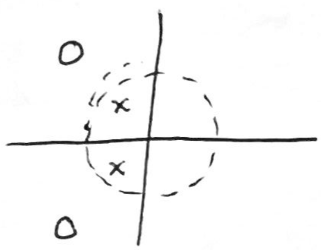


z_2 @ $-\frac{3}{2}e^{-j\pi/4}$ P_0 @ zero

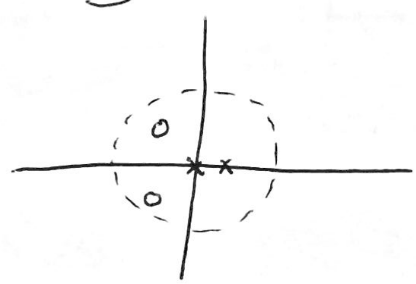
z_1 @ $-\frac{3}{2}e^{j\pi/4}$ P_1 @ $+\frac{1}{3}$

Add p-~~z~~-pairs @ $-\frac{2}{3}e^{-j\pi/4}$

and $-\frac{2}{3}e^{j\pi/4}$



Allpass

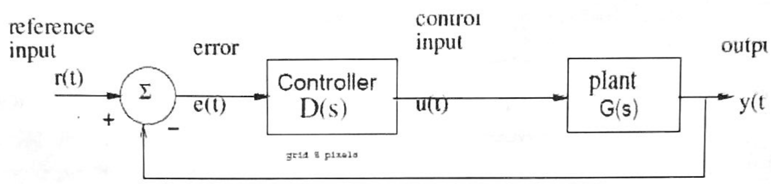


Min phase

$$\boxed{H_{Ap} = \frac{\left(1+\frac{3}{2}e^{j\pi/4}z^{-1}\right)\left(1+\frac{3}{2}e^{-j\pi/4}z^{-1}\right)}{\left(1+\frac{2}{3}e^{j\pi/4}z^{-1}\right)\left(1+\frac{2}{3}e^{-j\pi/4}z^{-1}\right)}\frac{4}{9} \qquad H_{mp} = \frac{\left(1+\frac{2}{3}e^{j\pi/4}z^{-1}\right)\left(1+\frac{2}{3}e^{-j\pi/4}z^{-1}\right)}{(z)\left(z-\frac{1}{3}\right)}\frac{9}{4}}$$

2. Steady-State Error From Arcak Lecture 22 (EE 120 Fall 2015)

Please refer to the "Steady State Error" Handout for some notes.

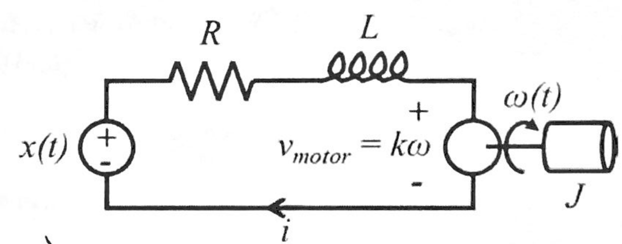


The "Type" of loop refers to the number of poles at $s = 0$ for the open loop gain, DG. Type 0 loops have constant steady state error for step response. Type 1 loops have zero steady state error for step response.

Exercise:

Consider the system below, where the input voltage $x(t)$ controls the angular velocity, $\omega(t)$, of the motor with moment of intertia J. The angular velocity is related to current by the diff. eq.: $J\frac{d\omega}{dt} = ki(t)$

a) Find $H_P(s)$, the transfer function from input voltage to output angular velocity.
b) Find the steady state error of the system under proportional control with gain K, and approximately sketch the steady state error vs. K.
c) What is the steady state error if we use integral control (K/s)?



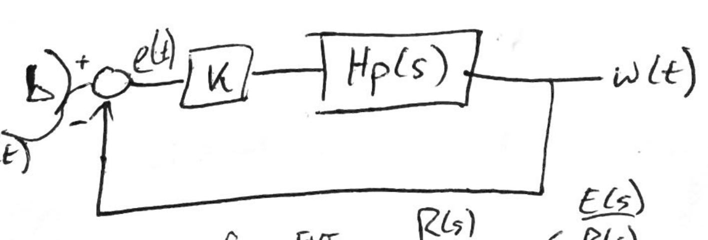
(Let $Y = \mathcal{L}\{\omega(t)\}$)

a) $J\frac{d\omega}{dt} = k i(t) \longleftrightarrow Js\,Y(s) = kI(s)$

$L\frac{di}{dt} = -k\omega(t) - Ri(t) + x(t) \longleftrightarrow LsI(s) = -kY(s) - RI(s) + X(s)$

Solve to get: $\boxed{\frac{Y(s)}{X(s)} = \frac{k}{JLs^2 + JRs + k^2}} = H_p(s)$

b)



$\lim_{t\to\infty} e(t) ? \overset{\text{FVT}}{=} \lim_{s\to 0} sE(s)$

$E(s) = R(s) - KH_p E(s)$

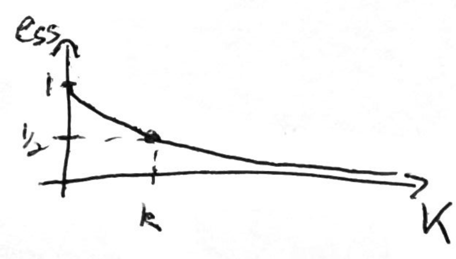
$E(s) = R(s)\frac{1}{1+KH_p(s)}$

Use $r(t) = u(t)$
$\rightarrow R(s) = \frac{1}{s}$

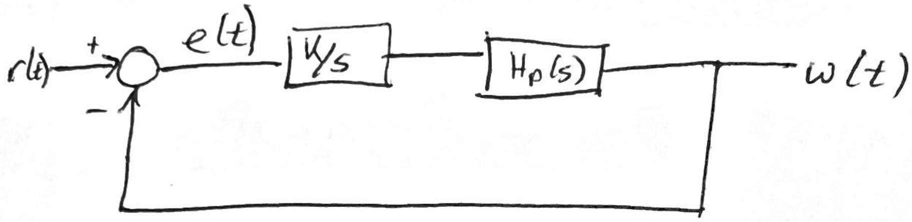
from FVT, $R(s)$, $\frac{E(s)}{R(s)}$

$\lim_{s\to 0} sE(s) = \lim_{s\to 0} (s)\left(\frac{1}{s}\right)\left(\frac{1}{1+KH_p(s)}\right)$

$H_p(s=0) = \frac{k}{k^2} = \frac{1}{k}$, $\lim_{s\to 0} sE(s) = \boxed{\frac{1}{1+\frac{K}{k}}}$



c) If we use integral controls $K \to \frac{K}{s}$



$$E(s) = R(s) - \frac{K}{s} H_p(s) E(s) \qquad \lim_{t \to \infty} e(t) = \lim_{s \to 0} s E(s)$$

$$E(s) = R(s) \frac{1}{1 + \frac{K}{s} H_p(s)} \qquad \text{Let } r(t) = u(t) \to R(s) = \frac{1}{s}$$

$$\lim_{s \to 0} s E(s) = \lim_{s \to 0} s \frac{1}{s} \frac{1}{1 + \frac{K}{s} H_p(s)} = \lim_{s \to 0} \frac{s}{s + K H_p(s)}$$

$$= \frac{0}{0 + \frac{K}{k}} = \boxed{0 = e_{ss}}$$

The Loop in part b is a Type 0 loop (DG has no poles @ s=0)

The loop in part c is a Type 1 loop (DG has 1 pole @ s=0)

Note: Type 0 has a constant error for a step input
Type 1 has a zero error for a step input.