DISCUSSION 2014.03.06

DFT:

Let $x[n]$ be a finite-length seq. of length N, i.e.

$$x[n]=0 \quad \text{outside of} \quad 0 \le n \le N-1$$

DFT: of $x[n]$ (denoted $X[k]$) is defined by

$$X[k] = \begin{cases} \sum_{n=0}^{N-1} x[n]\, e^{-j2\pi kn/N} & , k=0,1,\dots,N-1 \\ 0 & , \text{o.w.} \end{cases}$$

IDFT: $$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]\, e^{j2\pi kn/N} \qquad n=0,1,\dots,N-1$$

2 important facts:

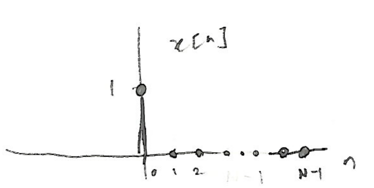
1. $\exists$ an extremely fast algorithm called the fast fourier transform (FFT) for DFT computation.
2. DFT is the appropriate Fourier representation for digital computer realization since it is discrete & of finite length in both the frequency & time domain.

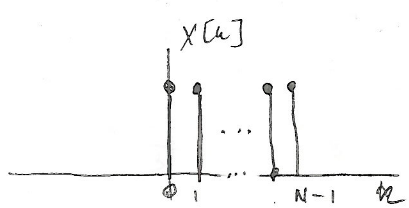
Examples

° Find the N-point DFT of the following sequences.

a) $x[n] = \delta[n]$

$$X[k] = \sum_{n=0}^{N-1} \delta[n]\, e^{-j2\pi kn/N} = 1 \quad , \quad k=0,\dots,N-1$$



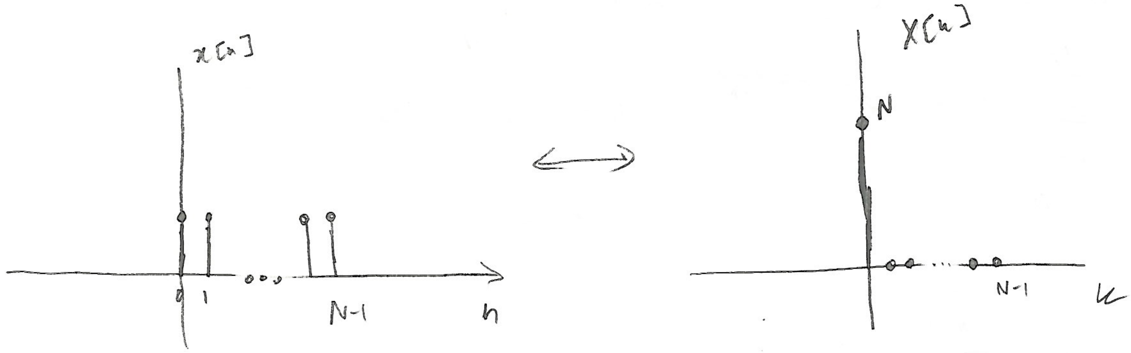


b) $x[n] = u[n] - u[n-N]$

$$X[k] = \sum_{n=0}^{N-1} e^{-jkn2\pi/N} = \frac{1 - e^{-jkN2\pi/N}}{1 - e^{-jk2\pi/N}} = 0 \quad \forall k \neq 0$$

Since $e^{-j2\pi kN/N} = e^{-j2\pi k} = 1$.

$$X[0] = \sum_{n=0}^{N-1} e^{-j2\pi \cdot 0/N} = \sum_{n=0}^{N-1} 1 = N$$



2° Consider two sequences $x[n]$; $h[n]$ of length 4 :

$$x[n] = \cos\left(\frac{\pi}{2} n\right) \qquad n = 0, 1, 2, 3$$

$$h[n] = \left(\frac{1}{2}\right)^n \qquad n = 0, 1, 2, 3$$

Find DFT of $y[n]$: from lecture we know $x_1[n] * x_2[n] \Longleftrightarrow X_1[k] X_2[k]$

$$X[k] = \sum_{n=0}^{3} x[n]\, e^{-j2\pi kn/4} = 1 - e^{-j4\pi k/4} = 1 - e^{-j\pi k}$$

$$H[k] = \sum_{n=0}^{3} h[n]\, e^{-j2\pi kn/4} = 1 + \frac{1}{2} e^{-j2\pi k/4} + \frac{1}{4} e^{-j4\pi k/4} + \frac{1}{8} e^{-6\pi k/4\, j}$$

$\frac{2k}{4}$

$\frac{1}{2} \quad \frac{4}{8} \quad \frac{1}{8}$

$\frac{3}{8}$

$-\frac{4}{8} + \frac{1}{8}$

$$Y[k] = X[k]\, H[k] = \left(1 - e^{-j\pi k}\right)\left(1 + \frac{1}{2} e^{-j\pi k/2} + \frac{1}{4} e^{-j\pi k} + \frac{1}{8} e^{-j3\pi k/2}\right)$$

$$= \left(\cancel{1} - e^{-j\pi k} + \frac{1}{2} e^{-j\pi k/2} - \frac{1}{2} e^{-3\pi k/2\, j} + \cancel{\frac{1}{4} e^{-j\pi k}} - \cancel{\frac{1}{4} e^{-2\pi k j}} + \cancel{\frac{1}{8} e^{-j3\pi k/2}} - \frac{1}{8} e^{-j5\pi k/2}\right)$$

$$= \frac{3}{4} - \frac{3}{4} e^{-j\pi k} + \frac{1}{2} e^{-j\pi k/2} - \frac{3}{8} e^{-j3\pi k/2} - \frac{1}{8} e^{-j5\pi k/2}$$

$$= \frac{3}{4} - \frac{3}{4} e^{-j\pi k} + \frac{3}{8} e^{-j\pi k/2} - \frac{3}{8} e^{-j3\pi k/2} \longrightarrow y[n] = \left\{\frac{3}{4}, \frac{3}{8}, -\frac{3}{4}, -\frac{3}{8}\right\}$$

Sampling

Let $x(t)$ be a real-valued band-limited signal s.t.

$$X(\omega) = 0 \quad , \quad |\omega| > \omega_M$$

See figs (a), (b)

Let $x_s(t)$ be defined by

$$x_s(t) = x(t)\,\delta_{T_s}(t) = x(t) \sum_{k=-\infty}^{\infty} \delta(t - kT_s)$$

a) Sketch $x_s(t)$ for $T_s < \pi/\omega_M$ & for $T_s > \pi/\omega_M$

We know that

$$x_s(t) = x(t)\,\delta_{T_s}(t) = x(t) \sum_{k=-\infty}^{\infty} \delta(t - kT_s)$$

$$= \sum_{k=-\infty}^{\infty} x(t)\,\delta(t - kT_s) = \sum_{k=-\infty}^{\infty} x(kT_s)\,\delta(t - kT_s)$$

see fig (e), (i) $\leftarrow$ $T_s > \pi/\omega_M$

& $T_s < \pi/\omega_M$

b) find & sketch the Fourier Spectrum $X_s(\omega)$ of $x_s(t)$ for $T_s < \pi/\omega_M$ & $T_s > \pi/\omega_M$

$$\delta_{T_s}(t) \longleftrightarrow \omega_s \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_s) \qquad \omega_s = \frac{2\pi}{T_s}$$

Then by freq. convolution thm.

$$X_s(\omega) = \mathcal{F}\{x(t)\,\delta_{T_s}(t)\} = \frac{1}{2\pi}\left(X(\omega) * \omega_s \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_s)\right)$$

$$= \frac{1}{T_s} \sum_{k=-\infty}^{\infty} X(\omega) * \delta(\omega - k\omega_s)$$

$$= \frac{1}{T_s} \sum_{k=-\infty}^{\infty} X(\omega - k\omega_s) \quad ☆$$

Ex. 2 Again consider band limited $x(t)$

$$X(\omega) = 0 \quad \text{for } |\omega| > \omega_m$$

WTS: $x(t)$ can be expressed as

$$x(t) = \sum_{k=-\infty}^{\infty} x(kT_s) \frac{\sin \omega_m (t - kT_s)}{\omega_m (t - kT_s)}, \qquad T_s = \frac{\pi}{\omega_m}$$

Soln: Let $x(t) \leftrightarrow X(\omega)$, $\quad x_s(t) = x(t)\,\delta_{T_s}(t) \longleftrightarrow X_s(\omega)$

Then from $\bigstar$, $\quad T_s X_s(\omega) = \sum_{k=-\infty}^{\infty} X(\omega - k\omega_s) \quad (\bigstar\bigstar)$

under the following conditions:

(1) $X(\omega) = 0$, $|\omega| > \omega_m$, $\quad$ (2) $T_s = \frac{\pi}{\omega_m}$

we have

$$X(\omega) = \frac{\pi}{\omega_m} X_s(\omega), \quad |\omega| < \omega_m \quad (\text{by } \bigstar\bigstar) \qquad (\boxed{\ })$$

Take the FT of

$$x_s(t) = \sum_{k=-\infty}^{\infty} x(kT_s)\,\delta(t - kT_s) \longleftrightarrow X_s(\omega) = \sum_{k=-\infty}^{\infty} x(kT_s)\, e^{-jkT_s\omega}$$

plug this eqn into $(\boxed{\ })$,

$$X(\omega) = \frac{\pi}{\omega_m} \sum_{k=-\infty}^{\infty} x(kT_s)\, e^{-jkT_s\omega} \qquad |\omega| < \omega_m$$

Take the IFT,

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, dt = \frac{1}{2\omega_m}\int_{-\omega_m}^{\omega_m} \sum_{k=-\infty}^{\infty} x(kT_s)\, e^{j\omega(t-kT_s)}\, d\omega$$

$$= \sum_{k=-\infty}^{\infty} x(kT_s) \frac{1}{2\omega_m}\int_{-\omega_m}^{\omega_m} e^{j\omega(t-kT_s)}\, d\omega$$

$$= \sum_{k=-\infty}^{\infty} x(kT_s) \frac{\sin(\omega_m(t-kT_s))}{\omega_m(t-kT_s)}$$

This says that a band limited signal can be recovered completely from a set of samples taken at a rate $f_s \geq 2f_m$. (uniform sampling theorem for low-pass signals)