EE120 Fall 2014 PS#8

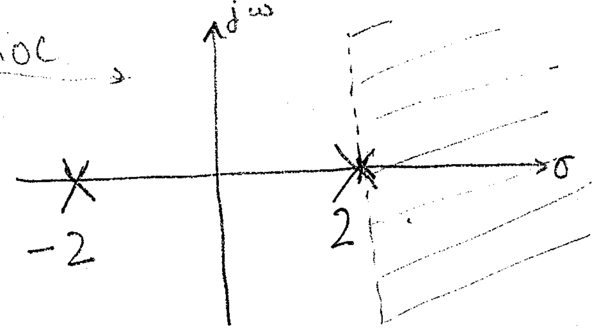
(1)

i) $x(t) = e^{-2t}u(t) \longleftrightarrow X(s) = \dfrac{1}{s+2}; \quad \sigma > -2$

$h(t) = e^{2t}u(t) \longleftrightarrow X(s) = \dfrac{1}{s-2}, \quad \sigma > 2$

$Y(s) = X(s)H(s)$ ROC

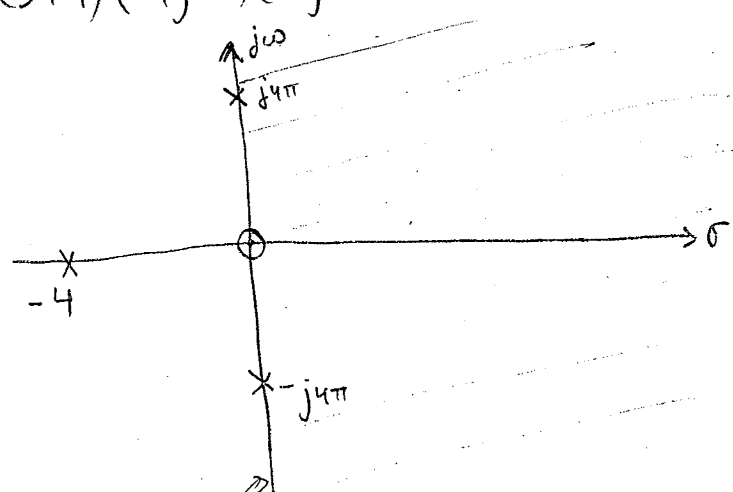
$= \dfrac{1}{s^2-4}$



ii) $x(t) = e^{-4t}u(t) \longleftrightarrow X(s) = \dfrac{1}{s+4}; \quad Re\{s\} > -4.$

$h(t) = \cos(4\pi t)u(t) \longleftrightarrow H(s) = \dfrac{s}{s^2+16\pi^2}; \quad Re\{s\} > 0$

$$Y(s) = \frac{s}{(s+4)(s+j4\pi)(s-j4\pi)}$$



jω-axis not in ROC

(2) a) $\frac{1}{3}(4M+m)L\ddot{\theta}(t) = (m+M)g\theta(t) - f(t)$

$$\frac{1}{3}(4M+m)L\left\{s^2\theta(s) - s\theta(0^-) - \dot{\theta}(0^-)\right\} = (m+M)g\theta(s) - F(s)$$

$$F(s) - \frac{1}{3}(4M+m)L\left[s\theta(0^-) + \dot{\theta}(0^-)\right] = \theta(s)\left[g(m+M) - \frac{1}{3}(4M+m)Ls^2\right]$$

$$\theta(s) = \frac{F(s) - \frac{1}{3}(4M+m)L\left[s\theta(0^-) + \dot{\theta}(0^-)\right]}{g(m+M) - \frac{1}{3}(4M+m)Ls^2} \longleftarrow ZIR$$

ZSR $\rightarrow$

b) $\underbrace{\frac{1}{3}(4M+m)L}_{B}\left\{s^2\theta(s) - s\theta(0^-) - \dot{\theta}(0^-)\right\} = (m+M)g\theta(s) - \alpha L\theta(s)$

$$= \theta(s)\left[\underbrace{(m+M)g}_{A} - \alpha L\right]$$

$A \leftarrow$ can be negative

$$\theta(s)\left[Bs^2 - A\right] = s\cdot B\cdot\theta(0^-) + B\dot{\theta}(0^-)$$

$$\theta(s) = \frac{s\cdot B\theta(0^-) + B\dot{\theta}(0^-)}{Bs^2 - A} = \frac{s\theta(0^-) + \dot{\theta}(0^-)}{s^2 - C}$$

where $C = \frac{A}{B}$

$s = \pm j\sqrt{C}$ or $s = \pm\sqrt{C}$

Note that $C \in \mathbb{R}$, so the poles of this will not both be in the left-half plane.

The system will thus not be stable and will not balance the broom. For imaginary pair, system will oscillate with constant amplitude.

c) $B\left\{s^2\theta(s) - s\theta(0^-) - \dot{\theta}(0^-)\right\} = (m+M)g\theta(s) - \alpha L\theta(s) - \beta L\left(s\theta(s) - \theta(0^-)\right)$

$$\theta(s)\left[Bs^2 - (m+M)g + \alpha L + \beta Ls\right] = sB\theta(0^-) + B\dot{\theta}(0^-) + \beta L\theta(0^-)$$

$$\theta(s) = \frac{sB\theta(0^-) + B\dot{\theta}(0^-) + \beta L\theta(0^-)}{Bs^2 + \beta Ls - (m+M)g + \alpha L}$$

Poles at $\dfrac{-\beta L \pm \sqrt{\beta^2L^2 - 4B(\alpha L - (m+M)g)}}{2B}$

When $\beta \leq 0$, poles are not both in left-half plane $\Rightarrow$ unstable

When $\beta > 0$, poles are both in left-half plane if

$$\beta^2L^2 - 4B(\alpha L - (m+M)g) < \beta^2L^2$$

$$4B(\alpha L - (m+M)g) > 0$$

$$\alpha L - (m+M)g > 0$$

$$\alpha > \frac{(m+M)g}{L}$$

So broom will be balanced when $\boxed{\alpha > \frac{(m+M)g}{L} \text{ and } \beta > 0}$

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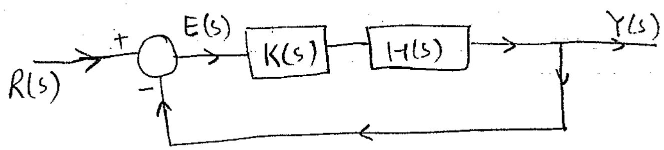
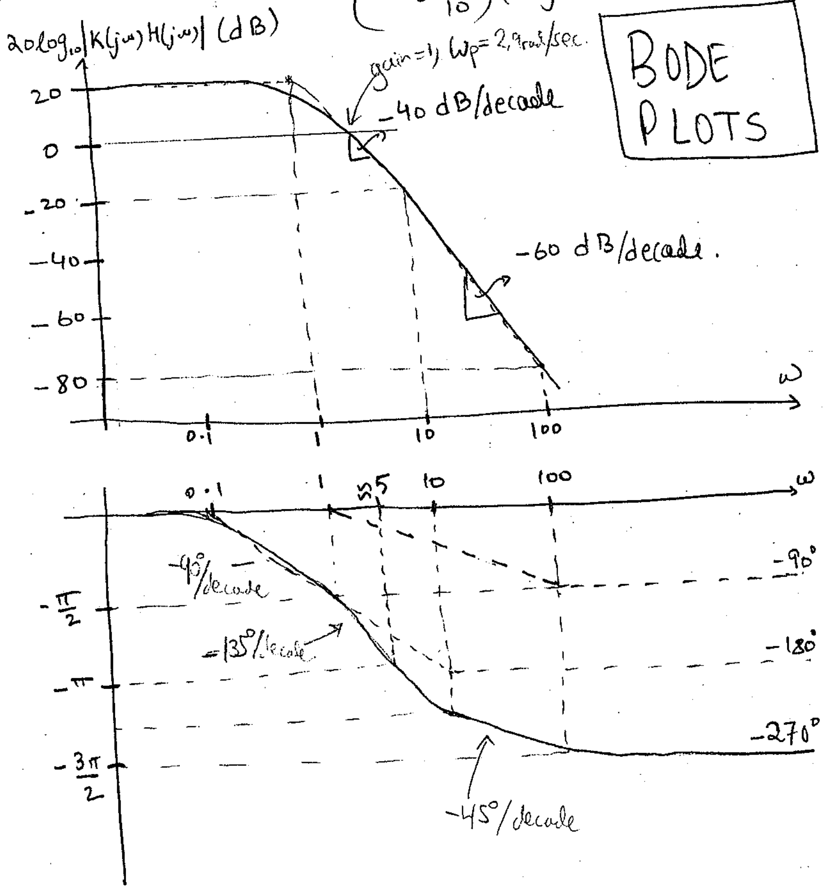
③ a)



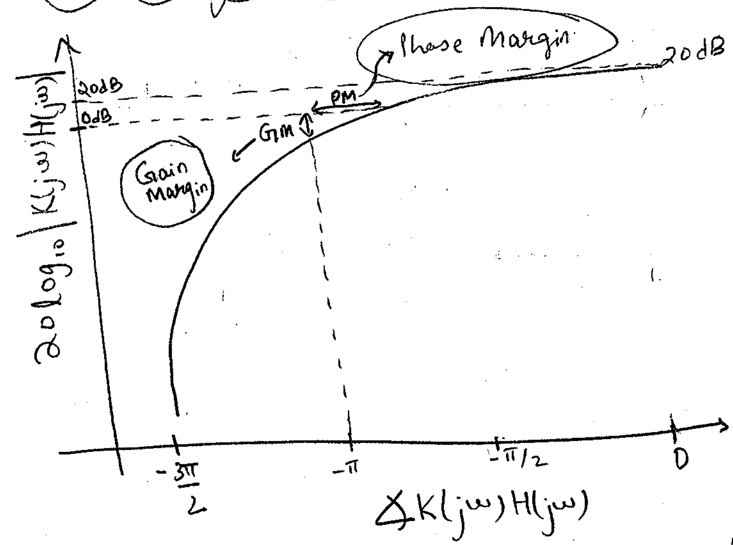
b) $K(j\omega)H(j\omega) = \dfrac{10}{\left(1+j\frac{\omega}{10}\right)(1+j\omega)^2}$

BODE PLOTS



$\measuredangle K(j\omega)H(j\omega) = -\tan^{-1}\left(\frac{\omega}{10}\right) - 2\tan^{-1}(\omega)$

Log Magnitude - Phase Diagram:



Gain & phase margins calculated in subsequent parts

→ This plot created by referring to the bode plots just shown.

→ Note the asymptote of $-\frac{3\pi}{2}$ in phase

3c) For finding exact phase, its okay to find it graphically using a plotting program, e.g Python / Matlab / Wolfram Alpha

Here is the formal mathematical evaluation:

$$\angle K(j\omega)H(j\omega) = -\tan^{-1}\left(\frac{\omega}{10}\right) - 2\tan^{-1}(\omega) = -\pi$$

$$\tan\left[\tan^{-1}\left(\frac{\omega}{10}\right) + 2\tan^{-1}(\omega)\right] = \tan(\pi) = 0.$$

$$\downarrow$$

$$\sin\left[\tan^{-1}\left(\frac{\omega}{10}\right) + 2\tan^{-1}(\omega)\right] = 0.$$

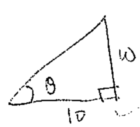
$$\sin\left(\tan^{-1}\left(\frac{\omega}{10}\right)\right)\cos\left(2\tan^{-1}(\omega)\right) + \cos\left(\tan^{-1}\left(\frac{\omega}{10}\right)\right)\sin\left(2\tan^{-1}(\omega)\right)$$

$$\frac{\omega}{\sqrt{\omega^2+100}}\left(2\left(\frac{1}{\sqrt{\omega^2+1}}\right)^2 - 1\right) + \frac{10}{\sqrt{\omega^2+100}} \times 2\frac{\omega}{\sqrt{\omega^2+1}} \times \frac{1}{\sqrt{\omega^2+1}} = 0$$

$$\omega\left(\frac{2}{\omega^2+1} - 1\right) + \frac{20\omega}{(\omega^2+1)} = 0$$

$$22\omega - \omega(\omega^2+1) = 0 \Rightarrow \omega^2 = 21$$

$\therefore$ $\boxed{\omega = 4.58 \text{ rad/s}}$ results in $\angle KH = -\pi$.



$$\cos\theta = \cos\left(\tan^{-1}\left(\frac{\omega}{10}\right)\right) = \frac{10}{\sqrt{\omega^2+100}}$$

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Now calculate gain margin:

$$|K(j\omega)H(j\omega)| = \frac{10}{\sqrt{1+\omega^2/100}\,(1+\omega^2)}$$

$$= \frac{10}{\sqrt{1+\frac{(4.58)^2}{100}}\,(1+(4.58)^2)} = 0.41$$

$\therefore$ Gain margin $= +20\log_{10}(0.41) = +7.7$ dB

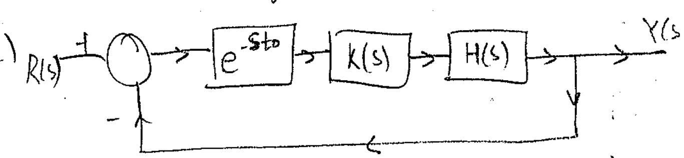
3d) $$\frac{10}{(1+\omega^2)\sqrt{1+\omega^2/100}} = 1 \rightarrow (1+\omega^2)^2\left(1+\frac{\omega^2}{100}\right) = 100$$

solution: $\omega = 2.93$ rad/s

$$\angle K(j\omega)H(j\omega) = -\tan^{-1}\left(\frac{2.93}{10}\right) - 2\tan^{-1}(2.93) = -2.77 \text{ rad}$$

$\therefore$ Phase margin $= \pi - 2.77 = 0.37$ rad.

3e)



In this case, $$\frac{Y(s)}{R(s)} = \frac{K(s)H(s)e^{-st_0}}{1-K(s)H(s)e^{-st_0}}$$

$$\angle K(j\omega)H(j\omega) = -\tan^{-1}\left(\frac{\omega}{10}\right) - 2\tan^{-1}(\omega) - \omega t_0$$

Found phase margin $= 0.37$ rad.

$$\therefore t_0 = \frac{0.37}{2.93} \Rightarrow \boxed{t_0 = 0.13 s}$$

maxim delay allowed while ensuring closed-loop stability

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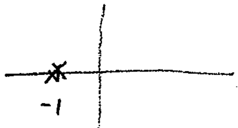
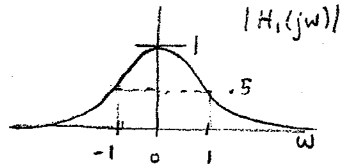
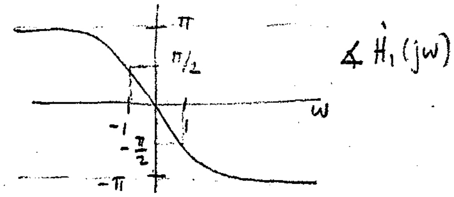
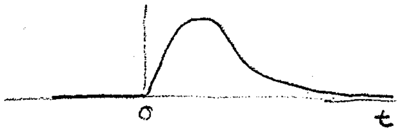
4 a)



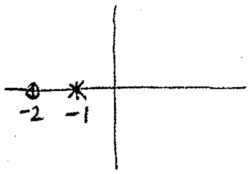


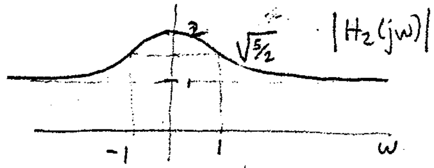


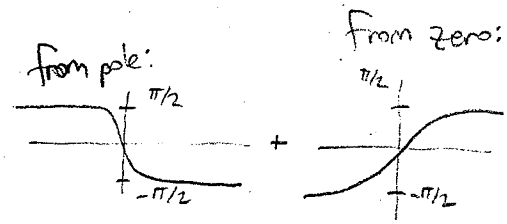
$H_1(s) = \frac{1}{(s+1)^2}$ $h_1(t) = te^{-t}u(t)$



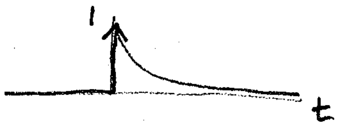
b)

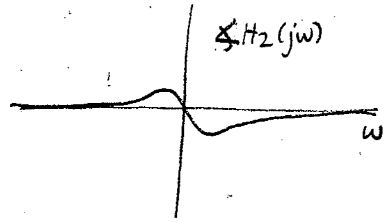




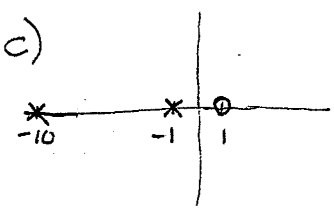


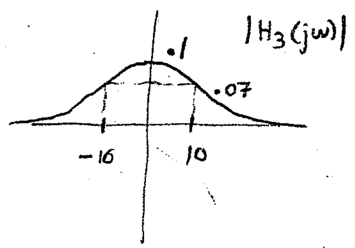
$H_2(s) = 1 + \frac{1}{s+1}$ $h_2(t) = \delta(t) + e^{-t}u(t)$

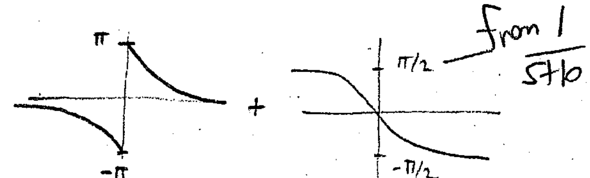




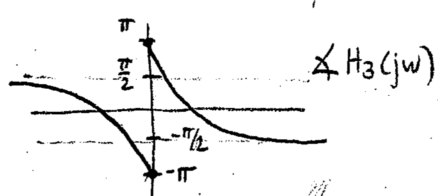
c)



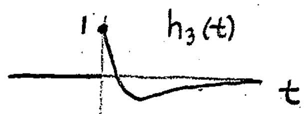




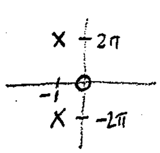
$H_3(s) = \underbrace{\frac{s-1}{s+1}}_{\text{all-pass}} \cdot \underbrace{\frac{1}{s+10}}_{\text{single-pole}} = \frac{11/9}{s+10} + \frac{-2/9}{s+1}$

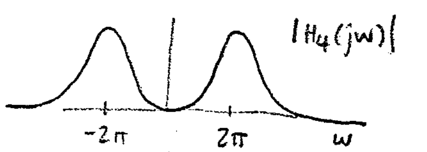


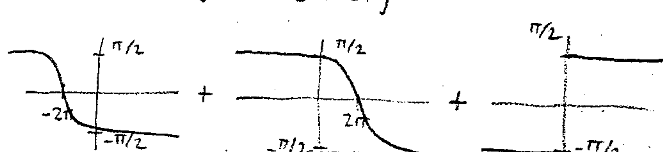
$h(t) = \frac{11}{9}e^{-10t}u(t) - \frac{2}{9}e^{-t}u(t)$



d) $H_4(s) = \frac{s}{s^2+2s+1+4\pi^2} = \frac{s}{(s+1-2\pi j)(s+1+2\pi j)} = \frac{\frac{1}{2}+j\frac{1}{4\pi}}{s+1-2\pi j} + \frac{\frac{1}{2}-j\frac{1}{4\pi}}{s+1+2\pi j}$





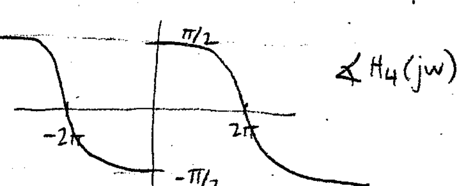


$h_4(t) = Ae^{(-1+2\pi j)t}u(t) + A^*e^{(-1-2\pi j)t}u(t)$

$= u(t)e^{-t} \cdot 2 \cdot Re\{Ae^{j2\pi t}\}$

$= u(t)e^{-t} \cdot 2 \cdot Re\{(\frac{1}{2}+j\frac{1}{4\pi})(\cos(2\pi t)+j\sin(2\pi t))\}$

$= u(t)e^{-t}\left[\cos(2\pi t) - \frac{1}{2\pi}\sin(2\pi t)\right]$



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⑤ for large w

a) Note ~~that~~ each pole causes $|H(j\omega)|$ to decrease $\propto 1/\omega$, while each zero causes $|H(j\omega)|$ to increase with ω. When there is an imbalance in the number of poles & zeros, @ $\omega \to \infty$ $|H(j\omega)|$ demonstrates either a net decreasing or a net increasing trend. In the provided magnitude response, $|H(\omega)|$ reaches a constant as $\omega \to \infty$, proof that the number of finite poles equal ~~numbers~~ of zeros.

Mathematically,

$$\lim_{s \to \infty} K \frac{\prod_{i=0}^{N}(s-a_i)}{\prod_{j=1}^{N}(s-b_j)} = K$$

b) $H(s) = \dfrac{s^K + a_1 s^{K-1} + a_2 s^{K-2} + a_3 s^{K-3} + \dots + a_K}{s^m + b_1 s^{m-1} + b_2 s^{m-2} + \dots + b_m}$

For real-time-domain signals, the coefficients $(a_1, \dots, a_K)$ and $(b_1, \dots, b_m)$ are real. This necessitates the poles and zeros to be purely real or appear as conjugate pairs.

Simple illustration: $s^3 + as^2 + bs + c = (s+\alpha)(s+\gamma+\beta j)(s+\gamma-\beta j)$ ensures real coefficients.

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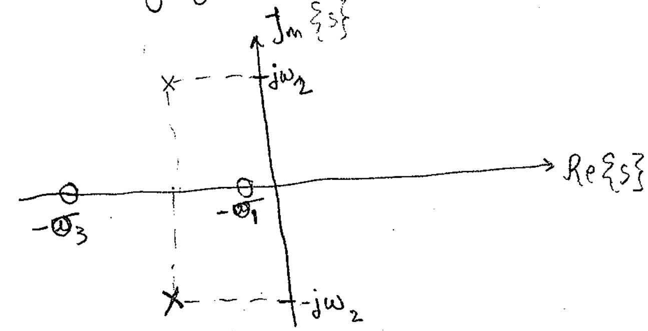
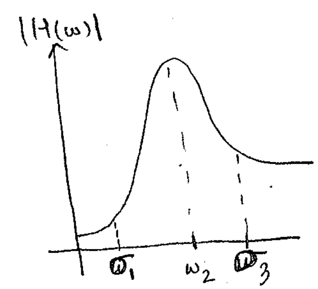
c) → $|H(\omega)|$ is flat near $\omega = 0$, so no poles/zeros at $\omega = 0$.

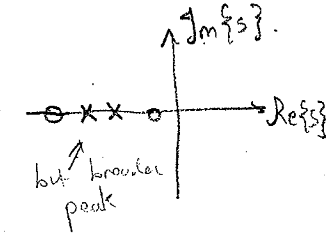
→ Peaking in $|H(\omega)|$ can be implemented with conjugate pole pairs.





* 1st zero (@ $-\sigma_1$) keeps low-freq gain small.
* Conjugate poles @ ω_2 contribute the peak.
* 2nd zero (@ $-\sigma_3$) contributes (+) slope to $|H|$, making $\lim_{\omega \to \infty} |H| \neq 0$.

Alternative p-z plot:



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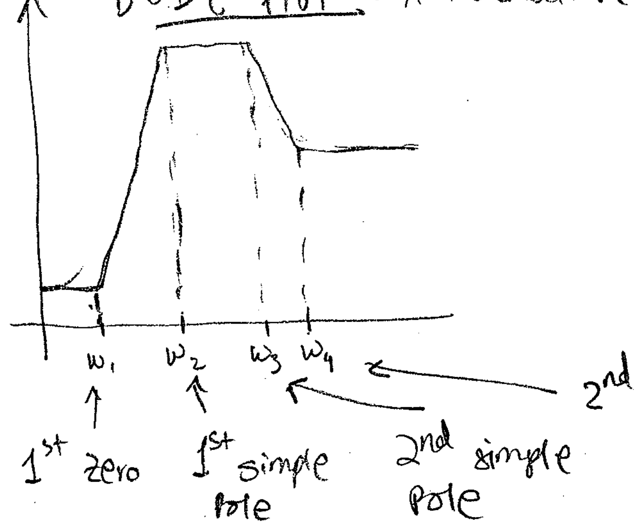
In this alternative, peak width is determined by separation between the two poles.

BODE PLOT: Alternative p-z map.



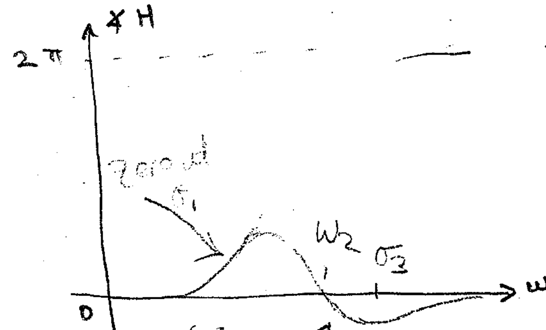
d) For the conjugate-pole system,

$$H(j\omega) = \frac{(j\omega + \omega_{z1})(j\omega + \omega_{z2})}{(j(\omega+\omega_1)-\sigma)(j(\omega-\omega_1)-\sigma)}$$

$$\measuredangle H = \tan^{-1}\left(\frac{\omega}{\omega_{z1}}\right) + \tan^{-1}\left(\frac{\omega}{\omega_{z2}}\right) - \tan^{-1}\left(\frac{\omega+\omega_1}{\sigma}\right) - \tan^{-1}\left(\frac{\omega-\omega_1}{\sigma}\right)$$

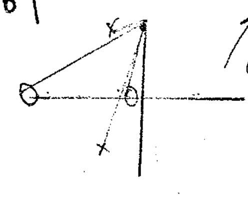
$$\measuredangle H\big|_{\omega=0} = 0$$

$$\measuredangle H\big|_{\omega\to\infty} = \frac{\pi}{2} + \frac{\pi}{2} - \frac{\pi}{2} - \frac{\pi}{2} = 0$$



* phase response not unique. Depends on the p-z map used; e.g., Zeros could be in RHP

angle contribution from poles > angle from zero(s)



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