EE120 HW #5 Fall 2014

① a) $\mathrm{Re}\{x[n]\} = \dfrac{x[n] + x^*[n]}{2}$

$$x^*[n] \xleftrightarrow{DTFT} \sum_{n=-\infty}^{\infty} x^*[n] e^{-j\omega n}$$

$$= \left(\sum_{n=-\infty}^{\infty} x[n] e^{j\omega n}\right)^*$$

$$= X^*(e^{-j\omega})$$

so $\mathrm{Re}\{x[n]\} \xleftrightarrow{DTFT} \frac{1}{2}X(e^{j\omega}) + \frac{1}{2}X^*(e^{-j\omega})$

b) $$x^*[-n] \xleftrightarrow{DTFT} \sum_{n=-\infty}^{\infty} x^*[-n] e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} x^*[n] e^{j\omega n} \quad \text{(change of variable)}$$

$$= \left(\sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}\right)^*$$

$$= X^*(e^{j\omega})$$

c) $\mathrm{Even}\{x[n]\} = \dfrac{x[n] + x[-n]}{2}$

$$x[-n] \xleftrightarrow{DTFT} \sum_{n=-\infty}^{\infty} x[-n] e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} x[n] e^{j\omega n} \quad \text{(change of variable)}$$

$$= X(e^{-j\omega})$$

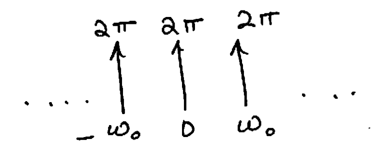
so $\mathrm{Even}\{x[n]\} \xleftrightarrow{DTFT} \frac{1}{2}X(e^{j\omega}) + \frac{1}{2}X(e^{-j\omega})$

②

$$x[n] = \sin\left(\frac{\pi n}{8}\right) - 2\cos\left(\frac{\pi n}{4}\right)$$

$$= \frac{1}{2j} e^{j\frac{\pi n}{8}} - \frac{1}{2j} e^{-j\frac{\pi n}{8}} - e^{j\pi n/4} - e^{-j\pi n/4}$$

$e^{j\omega_0 n} \longleftrightarrow$



$\therefore X(e^{j\omega})$: Periodic with 2π.

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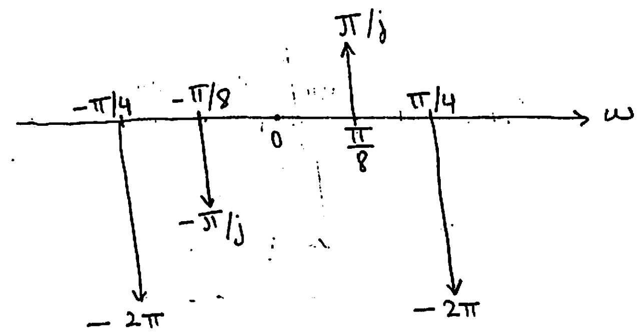
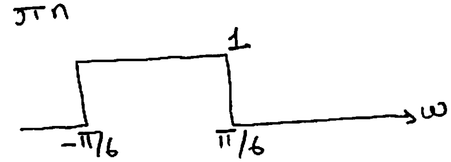
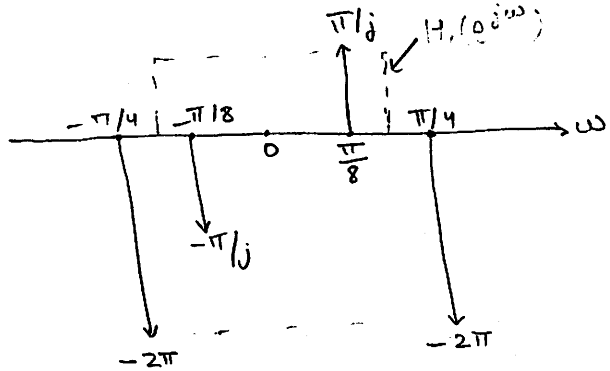


a) $h_1[n] = \dfrac{\sin(\pi n/6)}{\pi n}$

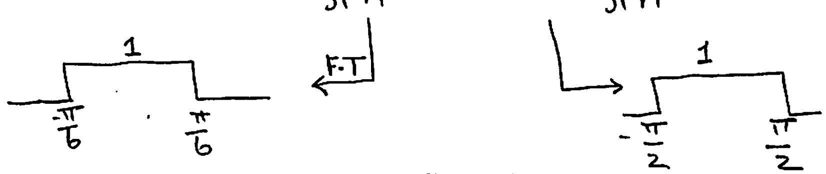
$H_1(e^{j\omega}) =$

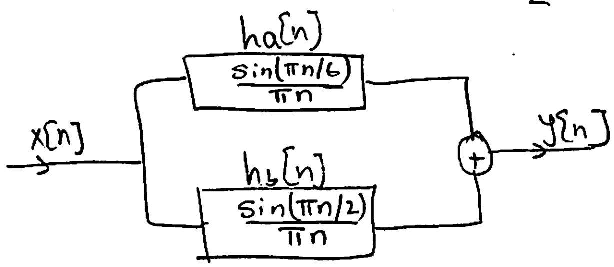




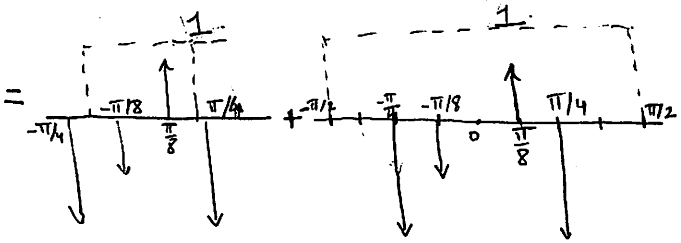
$\therefore \boxed{y[n] = \sin(\pi n/8)}$

b) $h_2[n] = \dfrac{\sin(\pi n/6)}{\pi n} + \dfrac{\sin(\pi n/2)}{\pi n}$





$X(e^{j\omega}) =$



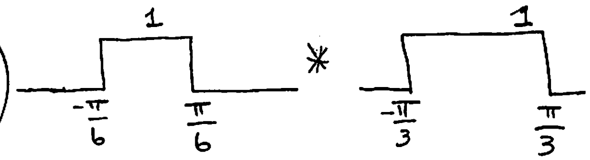
$\Downarrow \qquad\qquad\qquad \Downarrow$

$\sin\left(\frac{\pi n}{8}\right) \qquad\qquad \sin\left(\frac{\pi n}{8}\right) - 2\cos\left(\frac{\pi n}{4}\right)$

$\therefore \boxed{y[n] = 2\sin\left(\frac{\pi n}{8}\right) - 2\cos\left(\frac{\pi n}{4}\right)}$

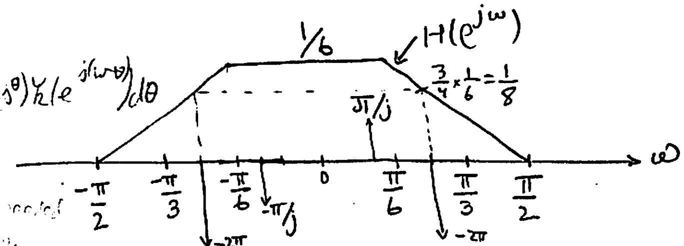
c) $h_3[n] = \dfrac{\sin(\pi n/6)}{\pi n} \; \dfrac{\sin(\pi n/3)}{\pi n}$

$H_3(e^{j\omega}) = \left(\frac{1}{2\pi}\right)$

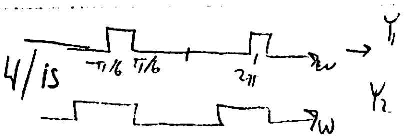


$H_3(e^{j\omega}) =$

$\frac{1}{2\pi}\int_{-\pi}^{\pi} Y_1(e^{j\theta}) Y_2(e^{j(\omega-\theta)})\, d\theta$



Note: circular convolution needed if overlap.



$$\therefore\ y[n] = \frac{1}{6}\sin\left(\frac{\pi n}{8}\right) - 2\cdot\left(\frac{1}{8}\right)\cos\left(\frac{\pi n}{4}\right)$$

$$\boxed{y[n] = \frac{1}{6}\sin\left(\frac{\pi n}{8}\right) - \frac{1}{4}\cos(\pi n/4)}$$

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③ a) $y[n] + \frac{1}{2}y[n-1] = x[n] - \frac{1}{4}x[n-1]$

$$Y(e^{jw}) + \frac{1}{2}Y(e^{jw})\cdot e^{-jw} = X(e^{jw}) - \frac{1}{4}X(e^{jw})\cdot e^{-jw}$$

$$Y(e^{jw})\left[1+\frac{1}{2}e^{-jw}\right] = X(e^{jw})\left[1-\frac{1}{4}e^{-jw}\right]$$

$$H(e^{jw}) = \frac{Y(e^{jw})}{X(e^{jw})} = \frac{1-\frac{1}{4}e^{-jw}}{1+\frac{1}{2}e^{-jw}}$$

$$G(e^{jw}) = \frac{1}{H(e^{jw})} = \frac{1+\frac{1}{2}e^{-jw}}{1-\frac{1}{4}e^{-jw}} = \frac{1}{1-\frac{1}{4}e^{-jw}} + \frac{1}{2}\cdot e^{-jw}\cdot\frac{1}{1-\frac{1}{4}e^{-jw}}$$

Since $a^n u[n] \longleftrightarrow \frac{1}{1-ae^{-jw}}$ and $\delta[n-1] \longleftrightarrow e^{-jw}$,

$$g[n] = \left(\tfrac{1}{4}\right)^n u[n] + \tfrac{1}{2}\,\delta[n-1] * \left(\tfrac{1}{4}\right)^n u[n]$$

$$g[n] = \left(\tfrac{1}{4}\right)^n u[n] + \tfrac{1}{2}\cdot\left(\tfrac{1}{4}\right)^{n-1} u[n-1]$$

$$\frac{Y(e^{jw})}{X(e^{jw})} = \frac{1+\frac{1}{2}e^{-jw}}{1-\frac{1}{4}e^{-jw}}$$

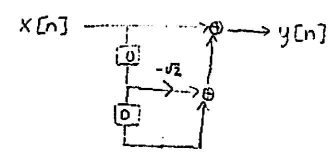
$$y[n] - \tfrac{1}{4}y[n-1] = x[n] + \tfrac{1}{2}x[n-1]$$

b) $$H(e^{jw}) = \frac{Y(e^{jw})}{X(e^{jw})} = \frac{e^{j2w} - \sqrt{2}\,e^{jw} + 1}{e^{2jw}}$$

$$Y(e^{jw})\cdot e^{j2w} = X(e^{jw})\cdot e^{j2w} - \sqrt{2}\,X(e^{jw})e^{jw} + X(e^{jw})$$

$$y[n+2] = x[n+2] - \sqrt{2}\,x[n+1] + x[n]$$

$$y[n] = x[n] - \sqrt{2}\,x[n-1] + x[n-2] \quad \text{(make causal)}$$



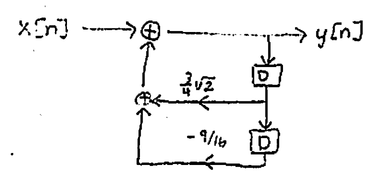
2 +'s
1 x
2 D's

c) $$H(e^{jw}) = \frac{Y(e^{jw})}{X(e^{jw})} = \frac{e^{j2w}}{e^{2jw} - \frac{3}{4}\sqrt{2}\,e^{jw} + 9/16}$$

$$y[n+2] - \tfrac{3}{4}\sqrt{2}\,y[n+1] + \tfrac{9}{16}y[n] = x[n+2]$$

$$y[n] - \tfrac{3}{4}\sqrt{2}\,y[n-1] + \tfrac{9}{16}y[n-2] = x[n]$$

$$y[n] = x[n] + \tfrac{3}{4}\sqrt{2}\,y[n-1] - \tfrac{9}{16}y[n-2]$$



2 +'s
2 x's
2 D's

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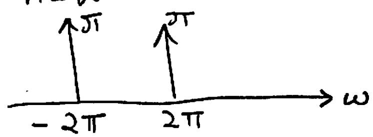
(4) $x(t) = \cos(2\pi t) = \frac{1}{2}e^{j2\pi t} + \frac{1}{2}e^{-j2\pi t}$

$$\tilde{x}(t) = x(t)\cdot \sum_{n=-\infty}^{n=\infty} \delta(t - n/8)$$

a) $X(j\omega) =$

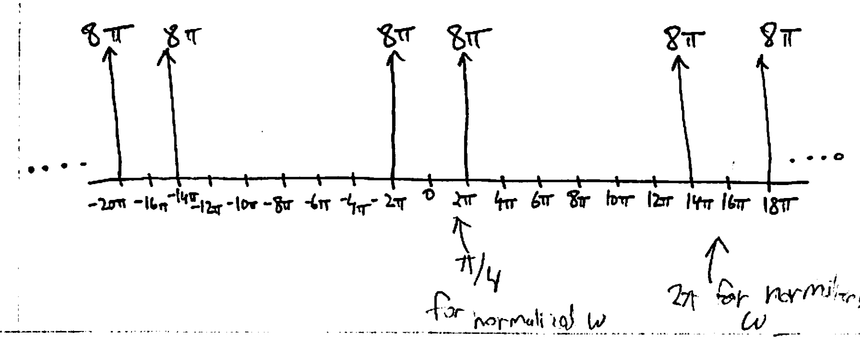


Define $p(t) = \sum_{n=-\infty}^{n=\infty} \delta\left(t - \frac{n}{8}\right)$

$$\therefore \tilde{X}(j\omega) = \frac{1}{2\pi} X(j\omega) * P(j\omega).$$

$$P(j\omega) = \frac{2\pi}{1/8} \sum_{k=-\infty}^{k=\infty} \delta(\omega - 16\pi k)$$

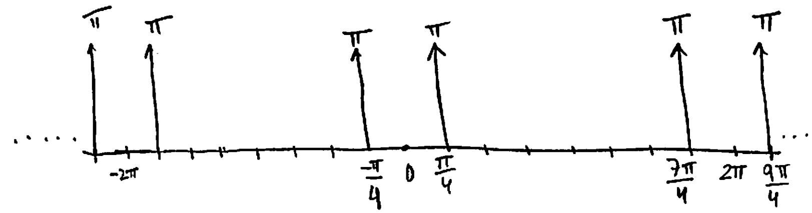
$\therefore \tilde{X}(j\omega)$:



b) $x[n] = \cos\left(\frac{\pi n}{4}\right) = \frac{1}{2}e^{j\pi n/4} + \frac{1}{2}e^{-j\pi n/4}$

$$e^{j\omega_0 n} \xleftrightarrow{DTFT} 2\pi \sum_{l=-\infty}^{\infty} \delta(\omega - \omega_0 - 2\pi l)$$

$$\therefore X(e^{j\omega}) = \pi \sum_{l=-\infty}^{\infty} \left[\delta\left(\omega - \frac{\pi}{4} - 2\pi l\right) + \delta\left(\omega + \frac{\pi}{4} - 2\pi l\right)\right]$$

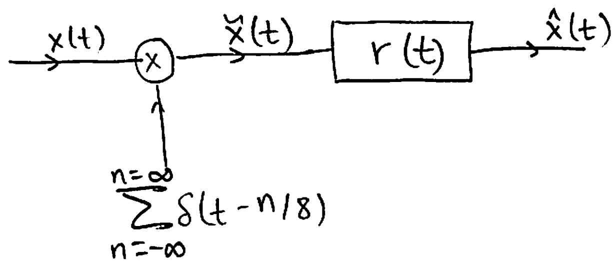


note that $X(e^{j\omega}) = T\tilde{X}(j\omega/T) = \frac{1}{8}\tilde{X}(j8\omega)$

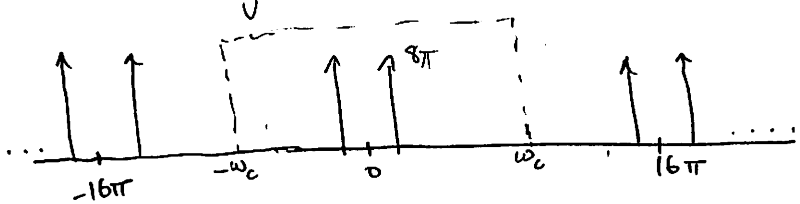
In relation of $X(j\omega)$,

$$X(e^{j\omega}) = X(8j\omega) * \sum_{k=-\infty}^{k=\infty} \delta(\omega - 2\pi k)$$

c)

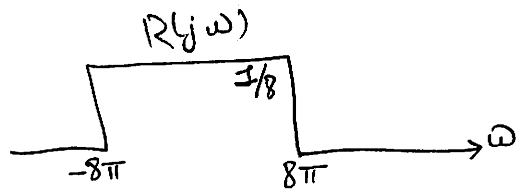


Recall $\check{X}(j\omega)$:



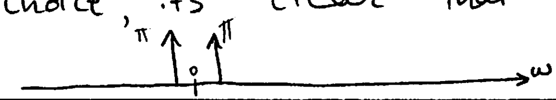
Pick an ideal LPF for $R(j\omega)$, where $2\pi < \omega_c < 14\pi$.

lets pick $\omega_c = 8\pi$.



With this choice, its clear that

$\hat{X}(j\omega) =$

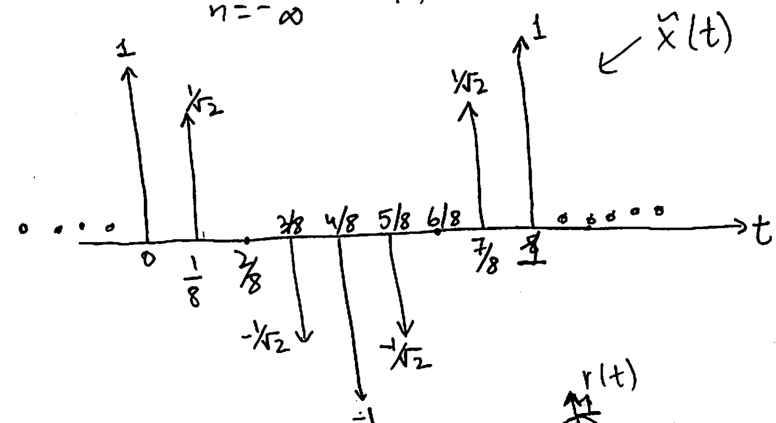


$\therefore \hat{x}(t) = x(t) = \cos 2\pi t$

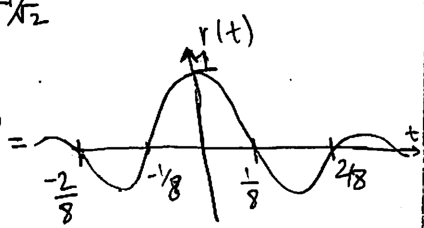
d) $\hat{x}(t) = \check{x}(t) * r(t)$

$$\check{x}(t) = \sum_{n=-\infty}^{n=\infty} x\left[\tfrac{n}{8}\right]\delta(t-n/8)$$

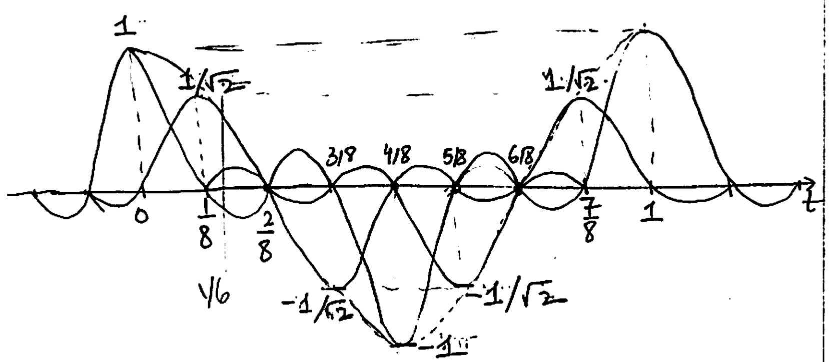
$$= \sum_{n=-\infty}^{n=\infty} \cos\left(\frac{\pi n}{4}\right)\delta(t-n/8)$$



$r(t) = \frac{1}{8}\frac{\sin(8\pi t)}{\pi t} =$



In order to convolve & estimate $\hat{x}(t = 1/6)$, put the sinc at all the impulses in $\check{x}(t)$.



Estimate $\hat{x}(t=1/6)$ by calculating the contribution from the main lobe centered at $t = 1/8$.

$$\hat{x}(t=1/6) = \frac{1}{8}\frac{1}{\sqrt{2}} \frac{\sin\left(8\pi\left(\frac{1}{6}-\frac{1}{8}\right)\right)}{\pi\left(\frac{1}{6}-\frac{1}{8}\right)} = 0.58$$

$$x(t=1/6) = \cos(2\pi/6) = 0.5$$

16% off! Improve estimate by adding contributions of first lobes due to the sinc centered @ $t=0$ & $t=3/8$.

$$\frac{1}{8}\frac{\sin(8\pi\cdot 1/6)}{\pi\cdot 1/6} = -0.21$$

Centered @ $t = 3/8$:

$$-\frac{1}{\sqrt{2}}\frac{1}{8}\frac{\sin(8\pi(\frac{1}{6}-3/8))}{\pi(1/6-3/8)} = +0.12$$

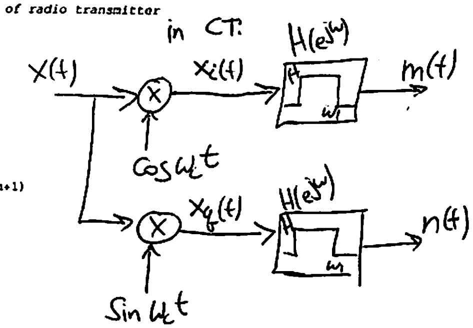
Now,

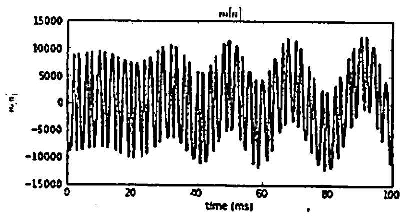
$$\hat{x}(t=1/6) = 0.58 - 0.21 + 0.12 = 0.49$$

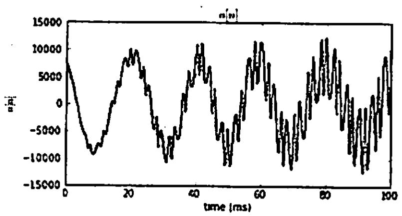
quite close to $\cos(2\pi/6)$.

```
[12]: # import file
      from scipy.io import wavfile
      rate,data= wavfile.read('xmit-signal.wav')  # 16 bit data from transmitter
      omega_c = 2.0 * np.pi * 5e5  # carrier frequency of radio transmitter
      print 'rate =', rate
      print 'data =', data
      length = np.size(data)
      print 'length  = ', length
      dt = 0.25*1e-6  # sample period
      time = dt * np.linspace(0,length,length)
      # set up low pass filter h[n]
      hlength = 48
      n = np.linspace(-hlength/2.0,hlength/2.0, hlength+1)
      h = np.sinc(n/8.0)/8.0
      # print 'h =', h

      inphase = np.cos(omega_c*time)*data
      quadrature = np.sin(omega_c*time)*data
      #print 'inphase', inphase[3950:4150]
      #print 'quadrature', quadrature[3950:4150]
      m_est = np.convolve(inphase,h)
      n_est= np.convolve(quadrature,h)

      fig = figure(figsize = (16,8))

      setup_graph(title='$m[n]$', x_label='time (ms)', y_label='$ m[n]$', fig_size=(6,3))
      _ = plt.plot(1000.0*time[0:length],m_est[0:length])
      setup_graph(title='$n[n]$', x_label='time (ms)', y_label='$ n[n]$', fig_size=(6,3))
      _ = plt.plot(1000.0*time[0:length],n_est[0:length])

      rate = 44100
      data = [  8852  16418  14377 ..., -21190 -14377    865]
      length  =  400000

      <matplotlib.figure.Figure at 0x1ff75ef0>
```

in CT:







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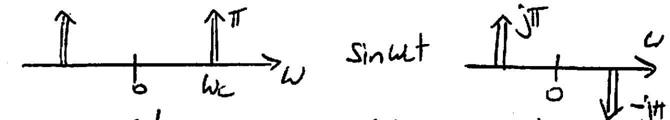
Problem 5:

$\cos\omega_c t$ $\quad$ $\sin\omega_c t$

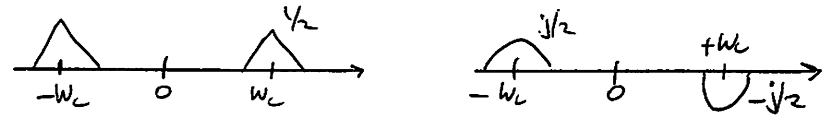


Assume $M(j\omega)$ $\quad$ $N(j\omega)$

(band limited, real & even for convenience)

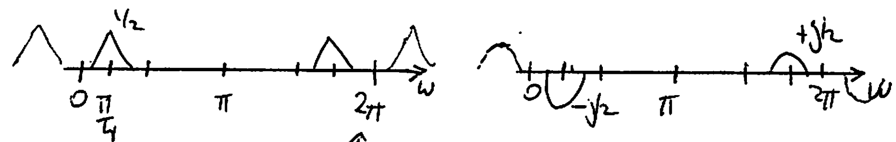
$$x(t) = m(t)\cos\omega_c t + n(t)\sin\omega_c t \longleftrightarrow X(j\omega) = \frac{1}{2}M(j\omega) * \left(\delta(\omega-\omega_c)+\delta(\omega+\omega_c)\right) + \frac{j}{2}N(j\omega) * \left(\delta(\omega+\omega_c)-\delta(\omega-\omega_c)\right)$$

$Re\{X(j\omega)\}$ $\quad$ $Im\{X(j\omega)\}$



Now sample to get DTFT, $\omega_s = 2\pi \cdot 4MHz = 8\omega_c$. Normalize ω:
~~let ω~~) $2\pi : 4MHz$, $\pi/4 : 500kHz$, $T_s = 0.25\mu s$

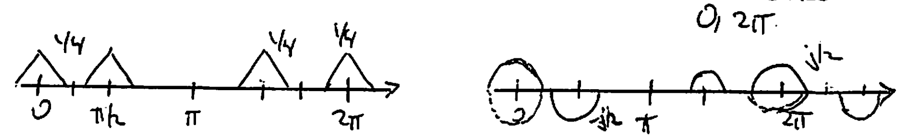
$Re\{X(e^{j\omega})\}$ $\quad$ $Im\{X(e^{j\omega})\}$



inphase

$$x_i[n] = x[n]\cdot\cos(\omega_c \cdot n \cdot T_s) \longleftrightarrow \frac{1}{2\pi}X(e^{j\omega}) * \pi\left(\delta(\omega+\omega_c)+\delta(\omega-\omega_c)\right)$$

$Re\{X_i(e^{j\omega})\}$ $\quad$ $Im\{X_i(e^{j\omega})\}$ $\quad$ note cancellation at 0, 2π.



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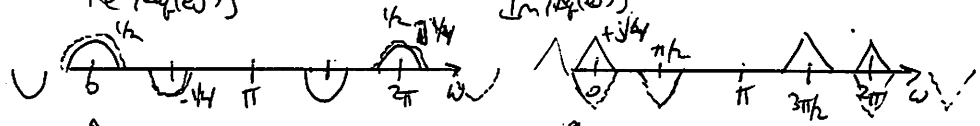
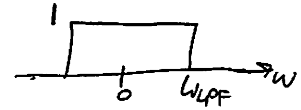
Quadrature $x_q[n] = x[n] \cdot \sin \omega_c n T_s \longleftrightarrow \frac{1}{2\pi} X(e^{j\omega}) * \frac{\pi j}{2}\left(\delta(\omega+\omega_c) - \delta(\omega-\omega_c)\right)$

$\mathrm{Re}\{X_q(e^{j\omega})\}$ $\quad\quad$ $\mathrm{Im}\{X_q(e^{j\omega})\}$



↑superposed at $\omega = 0$ $\quad\quad$ cancels at $0, 2\pi$

Need LPF $H(e^{j\omega})$ to reject image at $\omega = \pi/2$

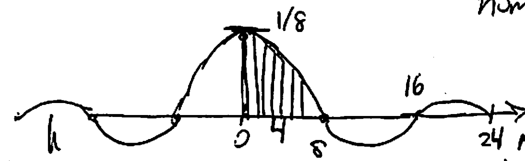


ω_{LPF} should be less than $\pi/2$.

for convenience, choose $\omega_c/2 = \pi/8$

$$h[n] = \frac{1}{2\pi}\int_{-\pi/8}^{\pi/8} 1 \cdot e^{+j\omega n}\, d\omega = \frac{1}{2\pi}\cdot\frac{2\sin n\pi/8}{n} = \frac{\sin n\pi/8}{\pi n}$$

numpy.sinc ⟶ $= \frac{1}{8}\,\mathrm{sinc}(n/8)$



$24T_s = 6\mu s$

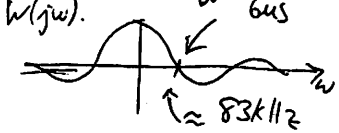
need to window in time $\quad h[n] \longrightarrow \otimes \longrightarrow h_w[n]$

$w[n]$

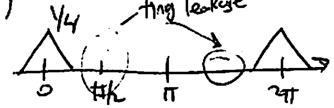
with rectangular window in CT

$$W(j\omega) = \int_{-6\mu s}^{6\mu s} e^{-j\omega t}\,dt = \frac{2\sin \omega \cdot 6\mu s}{\omega}$$

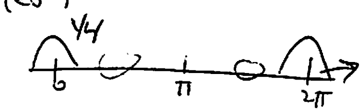
$W(j\omega)$. $\quad \omega = \frac{\pi}{6\mu s}$



$M(e^{j\omega})$



$N(e^{j\omega})$



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