EE120 HW #3 Fall 2014

P1 a) $z(t) = x(t)\,y(t)$

$$= \sum_k a_k e^{jk\omega_0 t} \sum_l b_l e^{jl\omega_0 t}$$

$$= \sum_k \sum_l a_k b_l e^{j(k+l)\omega_0 t} \qquad \text{let } m=k+l, \; l=m-k$$

$$= \sum_k \sum_m a_k b_{m-k} e^{jm\omega_0 t}$$

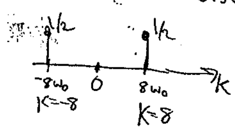
$$= \sum_m \underbrace{\left(\sum_k a_k b_{m-k}\right)}_{c_m} e^{jm\omega_0 t}$$

b) $x(t) = \cos(16\pi t)$

$$= \frac{1}{2}e^{j8\omega_0 t} + \frac{1}{2}e^{-j8\omega_0 t} \qquad \omega_0 = \frac{2\pi}{T_0} = 2\pi$$

$$= \sum a_k e^{jk\omega_0 t}$$

where $a_k = \begin{cases} \frac{1}{2}, & k=8 \text{ or } k=-8 \\ 0 & \text{else} \end{cases}$

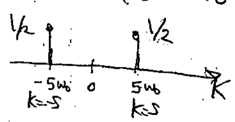


$y(t) = \cos(10\pi t)$

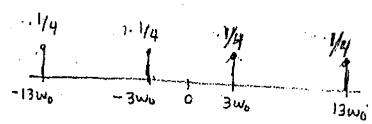
$$= \frac{1}{2}e^{j5\omega_0 t} + \frac{1}{2}e^{-j5\omega_0 t}$$

$$= \sum_k b_k e^{jk\omega_0 t}$$

where $b_k = \begin{cases} \frac{1}{2}, & k=5 \text{ or } k=-5 \\ 0 & \text{else} \end{cases}$



Note that c_k is given by the convolution of a_k and b_k, so the line spectra for $z(t) = x(t)y(t)$ is



$c_k = \begin{cases} \frac{1}{4}, & k=\pm 3, \pm 13 \\ 0 & \text{else} \end{cases}$

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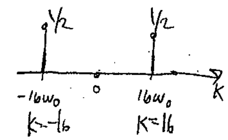
P1 c) $x(t) = \cos 16\pi t$

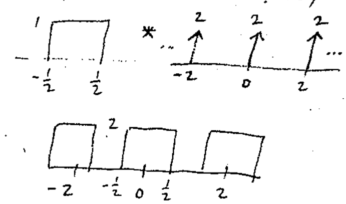
$$= \frac{1}{2}e^{j16\omega_0 t} + \frac{1}{2}e^{-j16\omega_0 t} \qquad \omega_0 = \frac{2\pi}{T_0} = \pi$$

$$= \sum a_k e^{jk\omega_0 t}$$

where $a_k = \begin{cases} \frac{1}{2}, & k=\pm 16 \\ 0 & \text{else} \end{cases}$



$y(t) = \Pi(t) * \text{comb}(t/2)$



$$b_k = \frac{2\sin(\pi k \cdot \text{duty\_cycle})}{\pi k}$$

$$= \frac{2\sin(\pi k/2)}{\pi k}$$

$b_0 = 1$

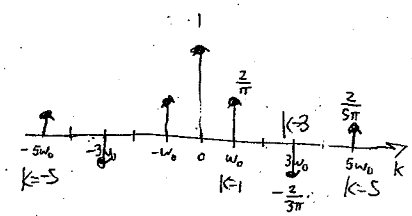
$b_1 = b_{-1} = \frac{2}{\pi}$

$b_2 = b_{-2} = 0$

$b_3 = b_{-3} = -\frac{2}{3\pi}$

$b_4 = b_{-4} = 0$

$b_5 = b_{-5} = \frac{2}{5\pi}$

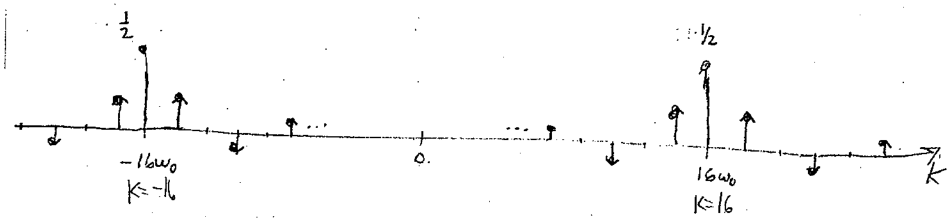


$$c_k = \sum_n a_n b_{k-n} = a_{16} b_{k-16} + a_{-16} b_{k+16}$$

$$= \frac{1}{2} \cdot \frac{2\sin\left(\frac{\pi}{2}(k-16)\right)}{\pi(k-16)} + \frac{1}{2} \cdot \frac{2\sin\left(\frac{\pi}{2}(k+16)\right)}{\pi(k+16)}$$

$$= \frac{\sin\left(\frac{\pi}{2}(k-16)\right)}{\pi(k-16)} + \frac{\sin\left(\frac{\pi}{2}(k+16)\right)}{\pi(k+16)}$$

line spectra



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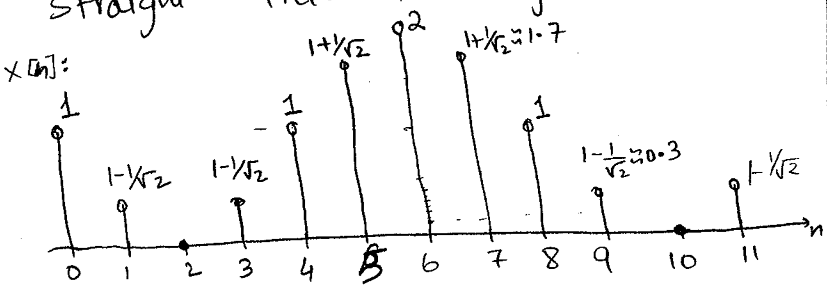
② a) $x[n] = 1 - \sin(n\pi/4)\ ;\quad 0 \le n \le 11$

$$a_k = \frac{1}{12}\sum_{n=0}^{n=11}\left[1-\sin\left(\frac{n\pi}{4}\right)\right]e^{-j\frac{2\pi}{12}kn}$$

$$a_k = \frac{1}{12}\sum_{n=0}^{n=11}\left[1-\sin\left(\frac{n\pi}{4}\right)\right]e^{-j\frac{\pi}{6}kn}$$

Easy approach: Visualize and understand the $x[n]$ before jumping straight into the Algebra.

$x[n]$:



*. Notice symmetry. If we plug this into the a_k equation, we get sums of conjugate complex exponentials, which we know turn into cosines.

I.O.W. a_k will be real

So: → start by plugging in $n=6$.

→ then, move on to values of n to the left & right of $n=6$.

→ note that because of the even symmetry, we expect the a_k to be sums of cosines.

$$a_k = \frac{1}{12}\sum_{n=0}^{11}\left[1-\sin\left(\frac{n\pi}{4}\right)\right]e^{-j\frac{\pi}{6}kn}$$

$$= \frac{1}{12}\Bigg[2e^{-j\pi k} + \left(1+\frac{1}{\sqrt{2}}\right)\underbrace{\left(e^{-j\frac{5\pi}{6}k} + e^{-j\frac{7\pi}{6}k}\right)}_{n=5,7} + \underbrace{\left(e^{-j\frac{2\pi}{3}k} + e^{-j\frac{4\pi}{3}k}\right)}_{n=4,8} + \left(1-\frac{1}{\sqrt{2}}\right)\underbrace{\left(e^{-j\frac{\pi}{2}k} + e^{-j\frac{3\pi}{2}k}\right)}_{n=3,9}$$

$$+ \left(1-\frac{1}{\sqrt{2}}\right)\underbrace{\left(e^{-j\frac{\pi}{6}k} + e^{-j\frac{11\pi}{6}k}\right)}_{n=1,11} + 1\Bigg]$$

Notice:

$$e^{-j\frac{5\pi}{6}k} + e^{-j\frac{7\pi}{6}k} = e^{-j\left(\pi-\frac{\pi}{6}\right)k} + e^{-j\left(\pi+\frac{\pi}{6}\right)k}$$

$$= e^{-jk\pi}\cdot 2\cos\left(\frac{\pi}{6}k\right)$$

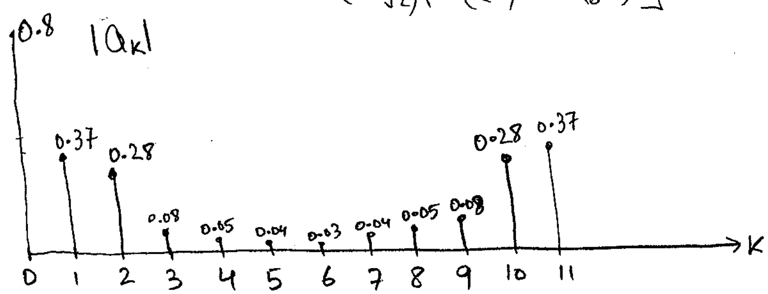
$$= (-1)^k\, 2\cos(\pi/6\, k)$$

Using this principle, simplify a_k:

2a

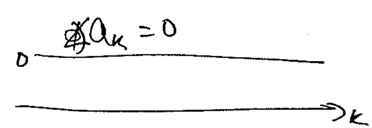
$$a_k = \frac{1}{12}\Big[1 + 2(-1)^k + 2(-1)^k \left(1+\frac{1}{\sqrt{2}}\right)\cos\left(\frac{\pi}{6}k\right) + 2(-1)^k \cos\left(\frac{\pi}{3}k\right) + \left(1-\frac{1}{\sqrt{2}}\right)\cos\left(\frac{\pi}{2}k\right)(2)(-1)^k$$
$$2(-1)^k\left(1-\frac{1}{\sqrt{2}}\right)\cos\left(\frac{5\pi}{6}k\right)\Big]$$

$$= \frac{1}{12}\Big[1 + 2(-1)^k\Big(1 + \left(1+\frac{1}{\sqrt{2}}\right)\cos\left(\frac{\pi}{6}k\right) + \cos\left(\frac{\pi}{3}k\right) + \left(1-\frac{1}{\sqrt{2}}\right)\left(\cos\left(\frac{\pi}{2}k\right) + \cos\left(\frac{5\pi}{6}k\right)\right)\Big)\Big]$$

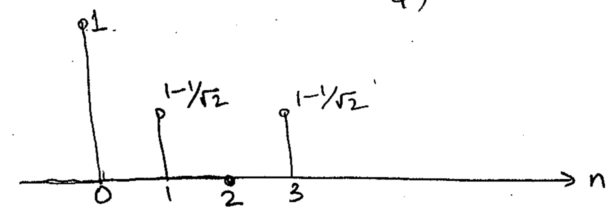


$a_1 = a_{11} = -0.37 \Rightarrow |a_1| = |a_{11}| = 0.37$

$$\boxed{\measuredangle a_k = 0}$$



(2) b) $N = 4$; $x[n] = 1 - \sin\left(\frac{\pi n}{4}\right)$; $0 \le n \le 3$.



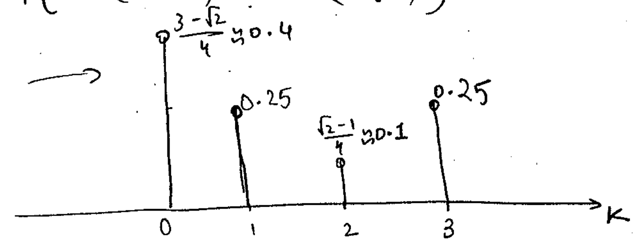
$$a_k = \frac{1}{4}\sum_{n=0}^{3} x[n]\, e^{-j\frac{2\pi}{4}kn}$$

$$a_k = \frac{1}{4}\left[1 + \left(1-\frac{1}{\sqrt{2}}\right)e^{-j\frac{\pi}{2}k} + \left(1-\frac{1}{\sqrt{2}}\right)e^{-j\frac{3\pi}{2}k}\right]$$

$$a_k = \frac{1}{4}\left[1 + \left(1-\frac{1}{\sqrt{2}}\right)e^{-j\frac{\pi}{2}k} + \left(1-\frac{1}{\sqrt{2}}\right)e^{-j(2\pi k - \pi k/2)}\right]$$

$$a_k = \frac{1}{4}\left[1 + \left(1-\frac{1}{\sqrt{2}}\right)2\cos\left(\frac{\pi}{2}k\right)\right]$$

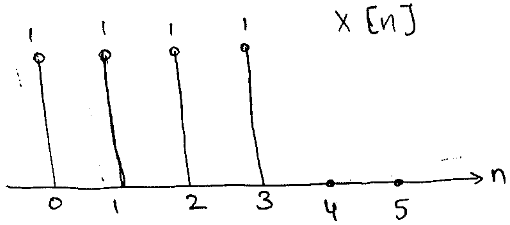
$|a_k| \longrightarrow$



$$\boxed{\measuredangle a_k = 0}$$

$1 - \frac{1}{\sqrt{2}} = 1 - \frac{\sqrt{2}}{2} = \frac{2-\sqrt{2}}{2}$

② c)



$$a_k = \frac{1}{6}\sum_{n=0}^{3} e^{-j\frac{2\pi}{6}kn} \; ; \; \boxed{a_0 = 2/3}$$

$$= \frac{1}{6}\,\frac{1-\left(e^{-j\frac{\pi}{3}k}\right)^4}{1-e^{-j\frac{\pi}{3}k}}$$

$$= \frac{1}{6}\,\frac{1-e^{-j\frac{4\pi}{3}k}}{1-e^{-j\frac{\pi}{3}k}}$$

$$\cdot\; |a_k|^2 = a_k a_k^*$$

$$= \frac{1}{36}\,\frac{1-e^{-j\frac{4\pi}{3}k}}{1-e^{-j\frac{\pi}{3}k}}\;\frac{1-e^{+j\frac{4\pi}{3}k}}{1-e^{+j\frac{\pi}{3}k}}$$

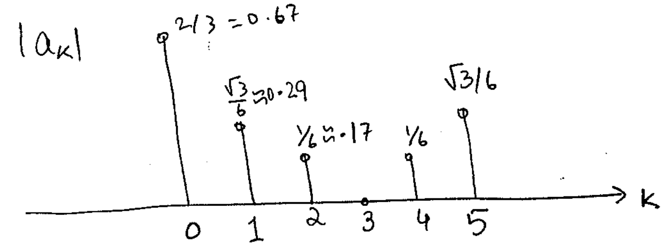
$$= \frac{1}{36}\,\frac{1-2\cos\left(\frac{4\pi}{3}k\right)+1}{2-2\cos\left(\frac{\pi}{3}k\right)}$$

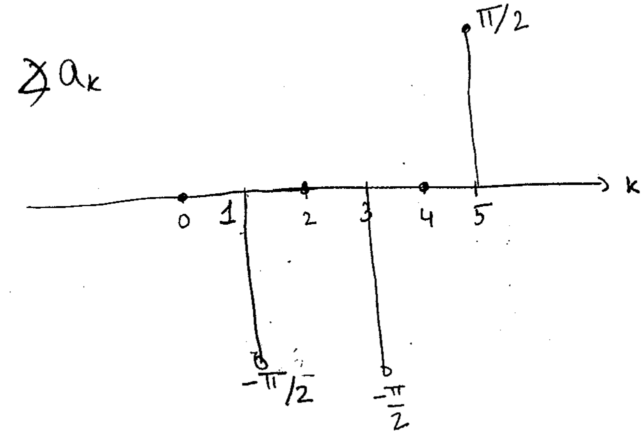
$$\therefore\; |a_k|^2 = \frac{1}{36}\,\frac{1-\cos\left(\frac{4\pi}{3}k\right)}{1-\cos\left(\frac{\pi}{3}k\right)}$$

$$|a_k| = \frac{1}{6}\sqrt{\frac{1-\cos(4\pi/3\,k)}{1-\cos(\pi k/3)}}$$

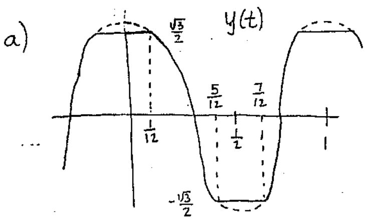
$$a_k = \frac{1}{6}\,\frac{\left(1-\cos\left(\frac{4\pi}{3}k\right)\right)+j\sin\left(\frac{4\pi}{3}k\right)}{\left(1-\cos\left(\frac{\pi}{3}k\right)\right)+j\sin\left(\frac{\pi}{3}k\right)}$$

$$\measuredangle a_k = \tan^{-1}\left(\frac{\sin(4\pi k/3)}{1-\cos(4\pi k/3)}\right) - \tan^{-1}\left(\frac{\sin\left(\frac{\pi}{3}k\right)}{1-\cos\left(\frac{\pi}{3}k\right)}\right)$$





(P3) a)



$T_0 = 1$

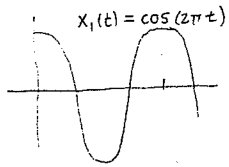
$\omega_0 = \frac{2\pi}{T_0} = 2\pi$

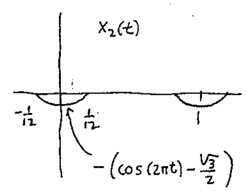
$\delta(t-1/2) \Leftrightarrow e^{-j\omega/2}$

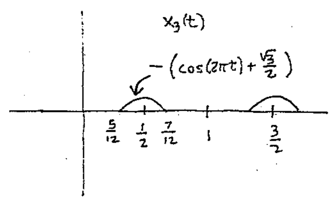
$x_3(t) = -\sum b_k e^{-jk\frac{2\pi}{2}} \cdot e^{+jk\omega_0 t}$

Note $x_3(t) = -x_2(t-1/2)$

$y(t) = x_1(t) + x_2(t) + x_3(t)$ where







$x_1(t) = \cos(2\pi t)$
$= \frac{1}{2}e^{j\omega_0 t} + \frac{1}{2}e^{-j\omega_0 t}$
$= \sum a_k e^{jk\omega_0 t}$

where $a_k = \begin{cases} \frac{1}{2}, & k = \pm 1 \\ 0 & \text{else} \end{cases}$

$x_2(t) = \sum b_k e^{jk\omega_0 t}$

$$b_k = \frac{1}{1}\int_{-\frac{1}{12}}^{\frac{1}{12}} \left[\frac{\sqrt{3}}{2} - \cos(2\pi t)\right] e^{-jk\omega_0 t}\, dt$$

$$= \frac{\sqrt{3}}{k2\pi}\sin\left(\frac{k\pi}{6}\right) - \frac{\sin\left((k-1)\frac{\pi}{6}\right)}{2\pi(k-1)} - \frac{\sin\left((k+1)\frac{\pi}{6}\right)}{2\pi(k+1)}$$

$x_3(t) = \sum c_k e^{jk\omega_0 t}$

$$c_k = -\frac{\sqrt{3}}{k2\pi}\sin\left(\frac{k\pi}{6}\right)(-1)^k + \frac{\sin\left((k-1)\frac{\pi}{6}\right)}{2\pi(k-1)}(-1)^k + \frac{\sin\left((k+1)\frac{\pi}{6}\right)}{2\pi(k+1)}(-1)^k$$

$$= (-1)^{k+1} \cdot b_k$$

so $y(t) = \sum d_k e^{jk\omega_0 t}$ where

$$d_k = a_k + b_k + c_k = \begin{cases} \frac{1}{3} + \frac{\sqrt{3}}{4\pi}, & k = \pm 1 \\ \frac{\sqrt{3}}{k\pi}\sin\left(\frac{k\pi}{6}\right) - \frac{\sin\left((k-1)\frac{\pi}{6}\right)}{\pi(k-1)} - \frac{\sin\left((k+1)\frac{\pi}{6}\right)}{\pi(k+1)}, & k = \pm 3, \pm 5, \ldots \\ 0 ; & k = 0, \pm 2, \pm 4, \ldots \end{cases}$$

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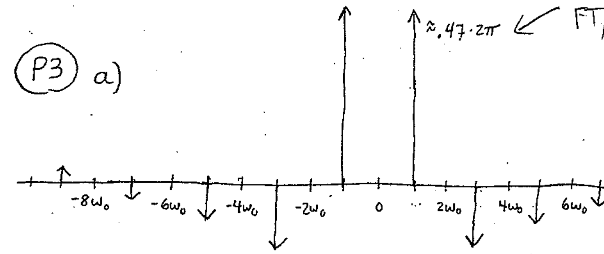
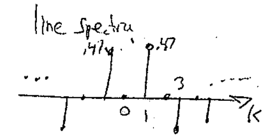
(P3) a)





b) $\text{Power} = \frac{1}{1}\int_0^1 y(t)^2\, dt$

$$= \left[\int_0^{1/12}\left(\frac{\sqrt{3}}{2}\right)^2 dt + \int_{1/12}^{5/12}\cos^2(2\pi t)\, dt + \int_{5/12}^{1/2}\left(-\frac{\sqrt{3}}{2}\right)^2 dt\right] \cdot 2$$ ← area from 1/2 → 1 is the same

$$= \left[\frac{3}{4}\cdot\frac{1}{12} + \int_{1/12}^{5/12}\frac{1+\cos(4\pi t)}{2}\, dt + \frac{3}{4}\cdot\frac{1}{12}\right]\cdot 2$$

$$= \frac{1}{4} + \int_{1/12}^{5/12} 1 + \cos(4\pi t)\, dt$$

$$= \frac{1}{4} + \left[t + \frac{1}{4\pi}\sin(4\pi t)\right]\Big|_{1/12}^{5/12}$$

$$= \frac{1}{4} + \frac{5}{12} + \frac{1}{4\pi}\sin\left(4\pi\cdot\frac{5}{12}\right) - \frac{1}{12} - \frac{1}{4\pi}\sin\left(4\pi\frac{1}{12}\right)$$

$$= \frac{7}{12} - \frac{\sqrt{3}}{4\pi} \approx .44550$$

c) Power in fundamental frequency $= |d_1|^2 + |d_{-1}|^2$

$$= 2\cdot\left(\frac{1}{3} + \frac{\sqrt{3}}{4\pi}\right)^2$$
$$= 2\left(\frac{1}{9} + \frac{\sqrt{3}}{6\pi} + \frac{3}{16\pi^2}\right)$$
$$= \frac{2}{9} + \frac{\sqrt{3}}{3\pi} + \frac{3}{8\pi^2}$$
$$\approx .44399$$

d) Power in higher harmonics $=$ Total power $- |d_0|^2 - |d_1|^2 - |d_{-1}|^2$

$$= \frac{7}{12} - \frac{\sqrt{3}}{4\pi} - \frac{2}{9} - \frac{\sqrt{3}}{3\pi} - \frac{3}{8\pi^2}$$
$$= \frac{13}{36} - \frac{7\sqrt{3}}{12\pi} - \frac{3}{8\pi^2}$$
$$\approx .00151$$

$$\text{THD} \approx \frac{.00151}{.44399} \approx .0034$$

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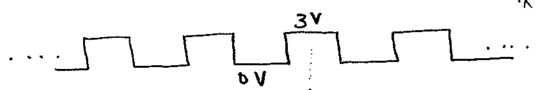
(4)

$x(t)$ $\quad a_k = \frac{1}{T_0}\int_{-\alpha T_0/2}^{\alpha T_0/2} e^{jk\omega_0 t}\,dt = \frac{\sin(k\pi\alpha)}{k\pi}$

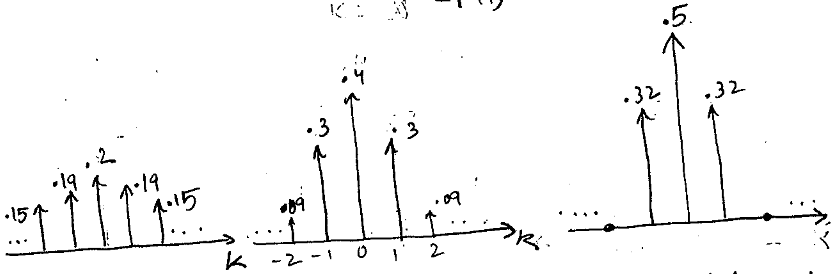


$$x(t) = \sum_{k=-\infty}^{k=\infty} a_k e^{jk\omega_0 n} \quad ; \quad \omega_0 = 2\pi\times 10^4 \tfrac{rad}{s}$$

$$e^{jk\omega_0 t} \longrightarrow \boxed{h(t)} \longrightarrow e^{jk\omega_0 t} H(jk\omega_0)$$

DC term on output $= a_0 H(j0)$

ripple term at 10kHz $= a_1 e^{j\omega_0 t} H(j\omega_0) + a_1 e^{-j\omega_0 t} H(-j\omega_0)$
$= r(t)$



20% duty-cycle $\qquad$ 40% duty-cycle $\qquad$ 50% duty-cycle

$$H(j\omega) = \int_0^{\infty} e^{-t/\tau} e^{-j\omega t}\,dt = \frac{\tau}{1+j\omega\tau}$$

$H(j0) = \tau$

$H(j\omega_0) = \frac{\tau}{1+j\omega_0\tau}$

$H(-j\omega_0) = \frac{\tau}{1-j\omega_0\tau} = H^*(j\omega_0)$

$r(t) = a_1\left[e^{j\omega_0 t}H(j\omega_0) + e^{-j\omega_0 t}H^*(j\omega_0)\right]$

let $H(j\omega_0) = be^{j\theta}$

$r(t) = a_1 b\left[e^{j\omega_0 t}e^{j\theta} + e^{-j\omega_0 t}e^{-j\theta}\right]$

4. cont. ripple at 10kHz $= 2a_1 b\cos(\omega_0 t + \theta)$

peak ripple $= 2a_1 b$ where $b = \left|\frac{\tau}{1+j\omega_0\tau}\right| = |H(j\omega_0)|$

$$|H(j\omega_0)| = \frac{\tau}{\sqrt{1+\omega_0^2\tau^2}} = \frac{10^{-2}}{\sqrt{1+(10^{-2})(2\pi\times10^4)^2}} = 1.59\times10^{-5} = b$$

$H(j0) = \tau = .01$, $\quad$ % ripple $= \frac{2a_1}{a_0}\frac{|H(j\omega_0)|}{H(j0)} = \frac{2a_1(1.59\times10^{-3})}{a_0}$

Average $= a_0$

a) 20% duty cycle $\quad a_0 = 0.2\,(3)$. $\quad a_1 = \frac{\sin 0.2\pi}{\pi} = .19\,(3)$ $\quad$, % ripple = 0.3%

b) 40% duty cycle $\quad a_0 = 0.4(3)$ $\quad a_1 = \frac{\sin 0.4\pi}{\pi} = .302\,(3)$ $\quad$ % ripple = 0.24%

c) 50% duty cycle $\quad a_0 = 0.5(3)$ $\quad a_1 = \frac{\sin 0.5\pi}{\pi} = 0.318\,(3)$ $\quad$ % ripple = 0.20%

(5) a) $x_1(t) = \cos(2\pi(t+\frac{1}{4\pi})) = \cos(2\pi t) * \delta(t+\frac{1}{4\pi})$

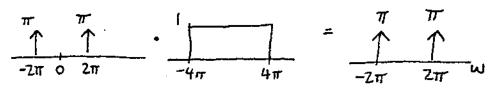
$h(t) = \frac{\sin(4\pi(t-1))}{\pi(t-1)} = \frac{\sin 4\pi t}{\pi t} * \delta(t-1)$

$$y_1(t) = x_1(t) * h(t) = \underbrace{\cos(2\pi t) * \frac{\sin(4\pi t)}{\pi t}}_{g(t)} * \underbrace{\delta(t+\tfrac{1}{4\pi}) * \delta(t-1)}_{\delta(t-1+\frac{1}{4\pi})}$$

$g(t) = g_1(t) * g_2(t)$

$\Updownarrow$

$G(j\omega) = G_1(j\omega) \cdot G_2(j\omega)$

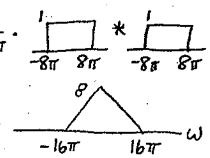


$\longleftrightarrow \cos(2\pi t)$

so $g(t) = \cos(2\pi t)$

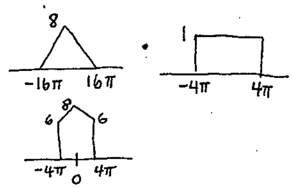
$$y_1(t) = \cos(2\pi t) * \delta(t-1+\tfrac{1}{4\pi}) = \cos(2\pi(t-1+\tfrac{1}{4\pi}))$$

b) $x_2(t) = \left(\frac{\sin 8\pi t}{\pi t}\right)\left(\frac{\sin 8\pi t}{\pi t}\right) \longleftrightarrow X_2(j\omega) = \frac{1}{2\pi} \cdot$

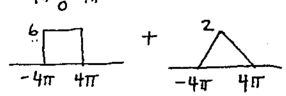


$$y_2(t) = x_2(t) * h(t) = \underbrace{x_2(t) * \frac{\sin 4\pi t}{\pi t}}_{g(t)} * \delta(t-1)$$

$g(t) \longleftrightarrow G(j\omega) =$



$g(t) = 6 \cdot \frac{\sin 4\pi t}{\pi t} + \left(\frac{\sin 2\pi t}{\pi t}\right)^2 \longleftrightarrow$



so $y_2(t) = 6 \cdot \frac{\sin 4\pi(t-1)}{\pi(t-1)} + \left(\frac{\sin 2\pi(t-1)}{\pi(t-1)}\right)^2$

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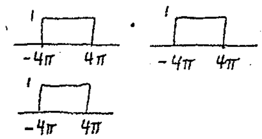
(5) c) $x_3(t) = \frac{\sin 4\pi(t+1)}{\pi(t+1)} = \frac{\sin 4\pi t}{\pi t} * \delta(t+1)$

$$y_3(t) = x_3(t) * h(t) = \frac{\sin 4\pi t}{\pi t} * \frac{\sin 4\pi t}{\pi t} * \underbrace{\delta(t+1) * \delta(t-1)}_{\delta(t)}$$

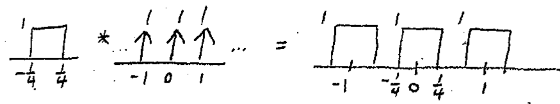
$g(t) \leftrightarrow G(j\omega) =$

$\frac{\sin 4\pi t}{\pi t} \longleftrightarrow$



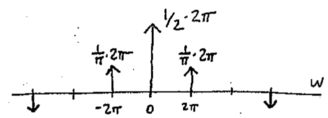
so $y_3(t) = \frac{\sin 4\pi t}{\pi t}$

d) $x_4(t) =$



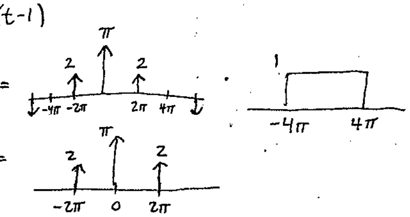
$= \sum_{k=-\infty}^{\infty} a_k e^{jk2\pi t}$ where $a_k = \frac{\sin(k\pi \cdot \text{duty\_cycle})}{k\pi} = \frac{\sin(k\pi/2)}{k\pi}$

$X_4(j\omega)$



$$y_4(t) = x_4(t) * h(t) = \underbrace{x_4(t) * \frac{\sin 4\pi t}{\pi t}}_{} * \delta(t-1)$$

$g(t) \longleftrightarrow G(j\omega) =$



$g(t) = \frac{1}{2} + \frac{2}{\pi}\cos(2\pi t) \longleftrightarrow$

so $y_4(t) = \frac{1}{2} + \frac{2}{\pi}\cos(2\pi(t-1))$

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