Second Order Circuits

- The series resonant circuit is one of the most important elementary circuits:

- The physics describes not only electrical LCR circuits, but also approximates mechanical resonance (mass-spring, pendulum, molecular resonance, microwave cavities, transmission lines, buildings, bridges, ...)

![Diagram of a series resonant circuit with inductor L, capacitor C, and resistor R connected in series.]
**Series LCR Impedance**

- With phasor analysis, this circuit is readily analyzed

\[
Z = j\omega L + \frac{1}{j\omega C} + R
\]

\[
Z = j\omega L + \frac{1}{j\omega C} + R = R + j\omega \left(1 - \frac{1}{\omega^2 LC}\right)
\]

\[
\text{Im}[Z] = \omega \left(1 - \frac{1}{\omega^2 LC}\right) = 0 \quad \omega^2 = \frac{1}{LC}
\]

**Resonance**

- Resonance occurs when the circuit impedance is purely real
- Imaginary components of impedance cancel out
- For a series resonant circuit, the current is maximum at resonance

\[
\omega < \omega_0 \quad \omega = \omega_0 \quad \omega > \omega_0
\]
Series Resonance Voltage Gain

- Note that at resonance, the voltage across the inductor and capacitor can be larger than the input voltage:

\[ V_L = I j\omega_0 L = \frac{V_S}{Z(\omega_0)} j\omega_0 L = \frac{V_S}{R} j\omega_0 L = jQ V_S \]

\[ V_C = I \frac{1}{j\omega_0 C} = \frac{V_S}{Z(\omega_0)} \frac{\omega_0 L}{j} = -\frac{V_S}{R} j\omega_0 L = -jQ V_S \]

\[ Q = \frac{\omega_0 L}{R} = \frac{1}{\omega_0 C} \frac{1}{R} = \frac{\sqrt{LC}}{1} \frac{1}{C} \frac{1}{R} = \frac{L}{C} \frac{1}{R} = \frac{Z_0}{R} \]

Second Order Transfer Function

- So we have:

\[ H(j\omega) = \frac{V_0}{V_S} = \frac{R}{j\omega L + \frac{1}{j\omega C} + R} \]

- To find the poles/zeros, let’s put the \( H \) in canonical form:

\[ H(j\omega) = \frac{V_0}{V_S} = \frac{j\omega CR}{1 - \omega^2 LC + j\omega RC} \]

- One zero at DC frequency → no DC current through a capacitor
Poles of 2nd Order Transfer Function

- Denominator is a quadratic polynomial:
  \[ H(j\omega) = \frac{V_0}{V_s} = \frac{j\omega CR}{1 - \omega^2 LC + j\omega RC} = \frac{j\omega R}{L} \]
  \[ = \frac{1}{\omega_0^2 + (j\omega)^2 + j\omega \frac{R}{L}} \]
  \[ = \frac{1}{\omega_0^2 + (j\omega)^2 + j\frac{\omega_0}{Q} \frac{R}{L}} \]
  \[ = \frac{1}{\omega_0^2 + (j\omega)^2 + j\frac{\omega_0}{Q}} \]

- Finding the poles...
  - Let's factor the denominator:
    \[ (j\omega)^2 + (j\omega) \frac{\omega_0}{Q} + \omega_0^2 = 0 \]
    \[ j\omega = -\frac{\omega_0}{2Q} \pm \sqrt{\left(\frac{\omega_0}{2Q}\right)^2 - \omega_0^2} = -\frac{\omega_0}{2Q} \pm j\omega_0 \sqrt{1 - \frac{1}{4Q^2}} \]
  - Poles are complex conjugate frequencies
  - The Q parameter is called the "quality-factor" or Q-factor
  - This is an important parameter:
    \[ Q \xrightarrow{R \to 0} \infty \]
### Resonance without Loss

- The transfer function can be parameterized in terms of loss. First, take the lossless case, $R=0$:

$$j\omega = \left(-\frac{\omega_0}{2Q} \pm \sqrt{\frac{\omega_0^2}{4Q^2} - \omega_0^2}\right)_{Q \to \infty} = \pm j\omega_0$$

- When the circuit is lossless, the poles are at real frequencies, so the transfer function blows up!
- At this **resonance** frequency, the circuit has zero imaginary impedance and thus zero total impedance
- Even if we set the source equal to zero, the circuit can have a steady-state response (oscillates)

### Magnitude Response

- The response peakiness depends on $Q$

$$H(j\omega) = \frac{j\omega \omega_0 R}{\omega_0 L} = \frac{j\omega \omega_0}{Q}$$

$$H(j\omega_0) = \frac{j\omega_0^2}{\omega_0^2 - \omega_0^2 + j\omega_0 \omega_0} = 1$$
**How Peaky is it?**

- Let's find the points when the transfer function squared has dropped in half:

  \[
  |H(j\omega)|^2 = \left(\frac{\omega - \omega_0}{Q}\right)^2 \\
  \left(\frac{\omega_0^2 - \omega^2}{\omega_0\omega/Q}\right) + \left(\frac{\omega - \omega_0}{Q}\right)^2 = \frac{1}{2}
  \]

- \[
  |H(j\omega)|^2 = \frac{1}{2} = \frac{1}{2}
  \]

  \[
  \left(\frac{\omega_0^2 - \omega^2}{\omega_0\omega/Q}\right) + 1
  \]

  \[
  \left(\frac{\omega_0^2 - \omega^2}{\omega_0\omega/Q}\right)^2 = 1
  \]

**Half Power Frequencies (Bandwidth)**

- We have the following:

  \[
  \left(\frac{\omega_0^2 - \omega^2}{\omega_0\omega/Q}\right)^2 = 1 \quad \rightarrow \quad \frac{\omega_0^2 - \omega^2}{\omega_0\omega/Q} = \pm 1
  \]

  \[
  \omega^2 + \frac{\omega_0\omega}{Q} - \omega_0^2 = 0
  \]

  \[
  \omega = \pm \frac{\omega_0}{2Q} \pm \sqrt{\frac{\omega_0^2}{4Q} - \omega_0^2} = \pm a \pm b \quad b > a \quad \rightarrow \quad a + b > 0 \quad -a + b > 0 \quad -a - b < 0
  \]

  Take positive frequencies:

  \[
  \Delta \omega = \omega_+ - \omega_- = \frac{\omega_0}{Q} \quad \Delta \omega = \frac{1}{Q}
  \]
**More “Notation”**

- Often a second-order transfer function is characterized by the “damping” factor as opposed to the “Quality” factor

\[
\omega_0^2 + (j\omega)^2 + j\frac{\omega_0^2}{Q} = 0
\]

\[
\tau = \frac{1}{\omega_0}
\]

\[
1 + (j\omega\tau)^2 + j\frac{\omega\tau}{Q} = 0
\]

\[
1 + (j\omega\tau)^2 + (j\omega\tau)2\zeta = 0
\]

\[
Q = \frac{1}{2\zeta}
\]

---

**Second Order Circuit Bode Plot**

- Quadratic poles or zeros have the following form:

\[
(j\omega\tau)^2 + (j\omega\tau)2\zeta + 1 = 0
\]

- The roots can be parameterized in terms of the damping ratio:

\[
\zeta = 1 \quad \Rightarrow \quad (j\omega\tau)^2 + (j\omega\tau)2 + 1 = (1 + j\omega\tau)^2
\]

\[
\zeta > 1 \quad \Rightarrow \quad (j\omega\tau)^2 + (j\omega\tau)2\zeta + 1 = (1 + j\omega\tau_1)(1 + j\omega\tau_2)
\]

\[
j\omega\tau = -\zeta \pm \sqrt{\zeta^2 - 1}
\]

*Two equal poles*

*Two real poles*
Bode Plot: Damped Case

- The case of $\zeta > 1$ and $\zeta = 1$ is a simple generalization of simple poles (zeros). In the case that $\zeta > 1$, the poles (zeros) are at distinct frequencies. For $\zeta = 1$, the poles are at the same real frequency:

$$\zeta = 1 \quad \Rightarrow \quad (j \omega \tau)^2 + (j \omega \tau)^2 + 1 = 1 + j \omega \tau^2$$

$$|1 + j \omega \tau|^2 = |1 + j \omega \tau|^2$$

$$20 \log |1 + j \omega \tau|^2 = 40 \log |1 + j \omega \tau|$$

$$\angle (1 + j \omega \tau)^2 = \angle (1 + j \omega \tau) + \angle (1 + j \omega \tau) = 2 \angle (1 + j \omega \tau)$$

Asymptotic Slope is 40 dB/dec

Asymptotic Phase Shift is 180°

Underdamped Case

- For $\zeta < 1$, the poles are complex conjugates:

$$(j \omega \tau)^2 + (j \omega \tau)2\zeta + 1 = 0$$

$$j \omega \tau = -\zeta \pm \sqrt{\zeta^2 - 1} = \zeta \pm j \sqrt{1 - \zeta^2}$$

- For $\omega \tau << 1$, this quadratic is negligible (0dB)
- For $\omega \tau >> 1$, we can simplify:

$$20 \log |j \omega \tau|^2 + (j \omega \tau)2\zeta + 1 \approx 20 \log |j \omega \tau|^2 = 40 \log |\omega \tau|$$

- In the transition region $\omega \tau \sim 1$, things are tricky!
The phase for the quadratic factor is given by:

\[
\angle ((j\omega \tau)^2 + (j\omega \tau)2\zeta + 1) = \tan^{-1}\left(\frac{2\omega \tau \zeta}{1-(\omega \tau)^2}\right)
\]

- For \( \omega \tau < 1 \), the phase shift is less than 90°
- For \( \omega \tau = 1 \), the phase shift is exactly 90°
- For \( \omega \tau > 1 \), the argument is negative so the phase shift is above 90° and approaches 180°
- Key point: argument shifts sign around resonance
Phase Bode Plot

Bode Plot Guidelines

- In the transition region, note that at the breakpoint:

\[(j\omega\tau)^2 + (j\omega\tau)2\zeta + 1 = (j)^2 + (2\zeta + 1) = 2\zeta = \frac{1}{Q}\]

- From this you can estimate the peakiness in the magnitude response.
- Example: For \(\zeta = 0.1\), the Bode magnitude plot peaks by 20 log(5) \(\sim\) 14 dB
- The phase is much more difficult. Note for \(\zeta = 0\), the phase response is a step function
- For \(\zeta = 1\), the phase is two real poles at a fixed frequency
- For \(0 < \zeta < 1\), the plot should go somewhere in between!
Energy Storage in “Tank”

- At resonance, the energy stored in the inductor and capacitor are

\[ w_L = \frac{1}{2} L i(t)^2 = \frac{1}{2} L I_m^2 \cos^2 \omega_0 t \]

\[ w_C = \frac{1}{2} C (v(t))^2 = \frac{1}{2} C \left( \frac{1}{C} \int i(\tau) d\tau \right)^2 \]

\[ = \frac{1}{2} C \frac{i_m^2}{\omega_0^2 C^2} \sin^2 \omega_0 t = \frac{1}{2} \frac{i_m^2}{\omega_0^2 C} \sin^2 \omega_0 t \]

\[ w_s = w_L + w_C = \frac{1}{2} I_m^2 (L \cos^2 \omega_0 t + \frac{1}{\omega_0^2 C} \sin^2 \omega_0 t) = \frac{1}{2} I_m^2 L \]

\[ W_{L,\text{max}} = W_S = \frac{1}{2} I_m^2 L \]

Energy Dissipation in Tank

- Energy dissipated per cycle:

\[ w_D = P \cdot T = \frac{1}{2} I_m^2 R \cdot \frac{2\pi}{\omega_0} \]

- The ratio of the energy stored to the energy dissipated is thus:

\[ \frac{w_s}{w_D} = \frac{\frac{1}{2} L I_m^2}{\frac{1}{2} I_m^2 R \cdot \frac{2\pi}{\omega_0}} = \frac{\omega_0 L}{R} \frac{1}{2\pi} = \frac{Q}{2\pi} \]
Physical Interpretation of Q-Factor

- For the series resonant circuit we have related the Q factor to very fundamental properties of the tank:

  \[ Q = 2\pi \frac{W_S}{W_D} \]

- The tank quality factor relates how much energy is stored in a tank to how much energy loss is occurring.

- If Q >> 1, then the tank pretty much runs itself ... even if you turn off the source, the tank will continue to oscillate for several cycles (on the order of Q cycles)

- Mechanical resonators can be fabricated with extremely high Q

Low-Noise Amplifier

- D. Shaeffer, T. Lee, ISSCC’97

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D. Shaeffer, T. Lee, ISSCC’97
thin-Film Bulk Acoustic Resonator (FBAR)

- Agilent Technologies (IEEE ISSCC 2001)
- Q > 1000
- Resonates at 1.9 GHz
- Cell phone duplexer

**Series LCR Step Response**

- Consider the transient response of the following circuit when we apply a step at input
- Without inductor, the cap charges with RC time constant (EECS 40)
- Where does the inductor come from?
  - Intentional inductor placed in series
  - Every physical loop has inductance! (parasitic)
**LCR Step Response: \( L \) Small**

- We know the steady-state response is a constant voltage of \( V_{dd} \) across capacitor (inductor is short, cap is open)
- For the case of zero inductance, we know solution is of the following form:

\[
v(t) = V_{dd} \left( 1 - e^{-t/\tau} \right)
\]

**LCR Circuit ODE**

- Apply KVL to derive governing dynamic equations:
  \[ v_s(t) = v_C(t) + v_R(t) + v_L(t) \]
- Inductor and capacitor currents/voltages take the form:
  \[
  i = i_C = C \frac{dv_C}{dt} \\
  v_L = L \frac{di}{dt} \\
  v_L = L \frac{d}{dt} \left( C \frac{dv_C}{dt} \right) = LC \frac{d^2v_C}{dt^2} \\
  v_R = iR = RC \frac{dv_C}{dt}
  \]
- We have the following 2\(^{nd}\) order ODE:
  \[
  v_s(t) = v_C(t) + RC \frac{dv_C}{dt} + LC \frac{d^2v_C}{dt^2}
  \]
Initial Conditions

- For the solution of a second order circuit, we need to specify the initial conditions (IC):
  \[ v_0(0) = V_C(0) = 0 \text{V} \]
  \[ i(0) = i_L(0) = 0 \text{V} \]

- For \( t > 0 \), the source voltage is \( V_{dd} \). We can now solve for the following non-homogeneous equation subject to above IC:
  \[ V_{dd} = v_C(t) + RC \frac{dv_C}{dt} + LC \frac{d^2v_C}{dt^2} \]

- Steady state:
  \[ \frac{d}{dt} \to 0 \]
  \[ V_{dd} = v_C(\infty) \]

Guess Solution!

- Let’s subtract out the steady-state solution:
  \[ v_C(t) = V_{dd} + v(t) \]
  \[ V_{dd} = \frac{d}{dt}v(t) + RC \frac{dv}{dt} + LC \frac{d^2v}{dt^2} \]
  \[ 0 = v(t) + RC \frac{dv}{dt} + LC \frac{d^2v}{dt^2} \]

- Guess solution is of the following form:
  \[ v(t) = Ae^{st} \]
  \[ 0 = Ae^{st} + RC(sAe^{st}) + LC(s^2Ae^{st}) \]
  \[ 0 = Ae^{st}\left(1 + RCs + LCs^2\right) \]
  \[ 0 = 1 + RCs + LCs^2 \]
Again We’re Back to Algebra

- Our guess is valid if we can find values of “s” that satisfy this equation:
  \[0 = 1 + RCs + LCS^2 \quad \rightarrow \quad 1 + (s\tau)2\zeta + (s\tau)^2 = 0\]
  \[Q = \frac{1}{2\zeta}\]
  \[\tau = \frac{1}{\omega_0}\]
- The solutions are: \[s\tau = -\zeta \pm \sqrt{\zeta^2 - 1}\]
- This is the same equation we solved last lecture!
- There we found three interesting cases:

  \[
  \zeta = \begin{cases} 
  < 1 & \text{Underdamped} \\
  = 1 & \text{Critically damped} \\
  > 1 & \text{Overdamped}
  \end{cases}
  \]

General Case

- Solutions are real or complex conj depending on if \(\zeta > 1\) or \(\zeta < 1\)

  \[s = \frac{1}{\tau}(-\zeta \pm \sqrt{\zeta^2 - 1}) = \begin{cases} 
  s_1 \\
  s_2
  \end{cases}
  \]

  \[v_C(t) = V_{dd} + A\exp(s_1 t) + B\exp(s_2 t)\]
  \[v_C(0) = V_{dd} + A + B = 0\]

  \[i(0) = C \frac{dv_C(t)}{dt} \bigg|_{t=0} = 0 \quad \Rightarrow \quad As_1\exp(s_1 t) + Bs_2\exp(s_2 t) \bigg|_{t=0} = 0\]

  \[As_1 + Bs_2 = 0\]
  \[A + B = -V_{dd}\]
Final Solution (General Case)

- Solve for $A$ and $B$:
  
  \[ A + \frac{s_1}{s_2}A = -V_{dd} \]
  
  \[ A = \frac{-V_{dd}}{1 - \frac{s_1}{s_2}} \]
  
  \[ B = -V_{dd} - A = \frac{-V_{dd}(1 - \frac{s_1}{s_2}) + V_{dd}}{1 - \frac{s_1}{s_2}} = \frac{s_1 V_{dd}}{1 - \frac{s_1}{s_2}} \]
  
  \[ V_C(t) = V_{dd} + \frac{-V_{dd}}{1 - \frac{s_1}{s_2}} \exp(s_1 t) + \frac{s_1 V_{dd}}{1 - \frac{s_1}{s_2}} \exp(s_2 t) \]
  
  \[ V_C(t) = V_{dd} \left( 1 - \frac{1}{1 - \frac{s_1}{s_2}} \left( e^{s_1 t} - \frac{s_1}{s_2} e^{s_2 t} \right) \right) \]

Overdamped Case

- $\zeta > 1$: Time constants are real and negative

\[ s = \frac{1}{\tau} (-\zeta \pm \sqrt{\zeta^2 - 1}) = \begin{cases} \frac{s_1}{s_2} < 0 \\
\end{cases} \]

\[ \tau = 1 \]

\[ V_{dd} = 1 \]

\[ \zeta = 2 \]
Critically Damped

- $\zeta > 1$: Time constants are real and equal
  
  \[ s = \frac{1}{\tau} (-\zeta \pm \sqrt{\zeta^2 - 1}) = -\frac{1}{\tau} \]
  
  \[ \lim_{\zeta \to 1} v_C(t) = V_{dd} \left( 1 - e^{-t/\tau} - te^{-t/\tau} \right) \]

Underdamped

- Now the $s$ values are complex conjugate
  
  \[ s_1 = a + jb \]
  \[ s_2 = a - jb \]
  
  \[ v_C(t) = V_{dd} + A \exp((a + jb)t) + B \exp((a - jb)t) \]
  
  \[ v_C(t) = V_{dd} + e^{at} \left( A \exp(jbt) + B \exp(-jbt) \right) \]
  
  \[ A^* = \frac{-V_{dd}}{1 - \frac{s_1}{s_2}} = \frac{V_{dd}}{s_2 - 1} = \frac{s_1 V_{dd}}{s_2} = B \]
  
  \[ v_C(t) = V_{dd} + e^{at} \left( A \exp(jbt) + A^* \exp(-jbt) \right) \]
**Underdamped (cont)**

- So we have:

\[
\begin{align*}
\nu_C(t) &= V_{dd} + e^{at} \left( A \exp(jbt) + A^* \exp(-jbt) \right) \\
\nu_C(t) &= V_{dd} + e^{at} \Re \left[ A \exp(jbt) \right] \\
\nu_C(t) &= V_{dd} + e^{at} 2|A| \cos(\omega t + \phi) \\
|A| &= \frac{V_{dd}}{1 + \frac{s_1}{s_2}} \\
\phi &= \angle \frac{V_{dd}}{1 + \frac{s_1}{s_2}}
\end{align*}
\]

**Underdamped Peaking**

- For \( \zeta < 1 \), the step response overshoots:

\[
\begin{align*}
\tau &= 1 \\
V_{dd} &= 1 \\
\zeta &= .5
\end{align*}
\]
Extremely Underdamped

\[ \tau = 1 \]
\[ V_{dd} = 1 \]
\[ \zeta = .01 \]