Lecture 17

• Last time:
  – Wrap-up two-port MOS amplifiers

• Today:
  – Sinusoidal signals
  – Phasor representation of sinusoids

Sinusoidal Function Review

\[ v(t) = V \cos(\omega t + \phi) \]

- amplitude (half of peak-to-peak)
- frequency (radian) \( \omega = 2\pi f = 2\pi (1/T) \)
- phase (degrees or radians)

Graphical Description

\[ v_1(t) = V \cos(\omega t) \]
\[ v_2(t) = V \cos(\omega t - 45) \]
\[ \omega = \frac{2\pi}{T} \]

Why are Sinusoids Important?

- Any periodic signal \( v(t) \) can be expressed as a sum of sinusoidal signals by a Fourier series expansion (EECS 20N, EE 120)
- The response of a linear circuit to a sinusoidal input, as a function of its frequency \( \omega \), leads to insights into the behavior of the circuit.
Linear Circuits

- **Theorem**: solutions for voltages and currents in a linear circuit (i.e., one consisting of \(R, L, C\) and dependent sources \(G_m, R_m, A_v,\) and \(A_i\)) with a *sinusoidal* signal as the input are:

\[
\begin{align*}
\text{RC Circuit with Sinusoidal Input} \\
&\quad v(t) = V_s \cos(\omega t) :\text{ set phase of source to zero (use as the reference)} \\
&\quad v_c(t) = V_c \cos(\omega t + \phi) :\text{ solution is a sinusoidal signal with the same frequency, but with a different amplitude and phase-shifted with respect to the source}
\end{align*}
\]

A Better Technique

- It is much more efficient to work with *imaginary exponentials* as "representing" sinusoids, since these functions are direct solutions of linear differential equations:

\[
\frac{d}{dt}(e^{j\omega t}) = j\omega(e^{j\omega t})
\]

- Note that EEs use \(j = (-1)^{1/2}\) rather than \(i\), since the symbol \(i\) is already taken for current

Using Imaginary Exponentials

\[
RC \frac{dv_c}{dt} + v_c = v_s
\]

Substitute:

\[
\begin{align*}
&v_s(t) = v_s e^{j\omega t} \\
&v_c(t) = v_c e^{j(\omega t + \phi)}
\end{align*}
\]

Result:

\[
\tau(j\omega)v_c e^{j(\omega t + \phi)} + v_c e^{j(\omega t + \phi)} = v_s e^{j\omega t}
\]
Finding the Amplitude Ratio

\[ t(j\omega)v_c e^{i\phi} + v_s e^{i\phi} e^{j\omega t} = v_s e^{i\omega t} \]

\[ j\omega v_c e^{i\phi} + e^{i\phi} v_c = v_s \]

... use to find amplitude and phase

Amplitude Ratio:

\[ \frac{v_c}{v_s} = \frac{1}{j\omega t e^{i\phi} + e^{i\phi}} = \frac{e^{-j\phi}}{1 + j\omega t} \]

Answer is a real number, so take magnitude

\[ \frac{v_c}{v_s} = \frac{1}{\sqrt{1 + (\omega t)^2}} \]

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Graphical Result for Amplitude Ratio

Deviations:

- 1

- 1/10

- 1/10/τ

- 1/τ

- 1/10/τ

- 1.0

- 0.5

- 0.707

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Amplitude: a new representation

- We are interested in very small ratios (e.g., \( V_c/V_s = 0.0001 \))

- Therefore, we use a log plot … but we also define a new function called the deciBel (after Alex. Graham Bell)

\[ (V_c/V_s)_\text{dB} = 20 \log_{10} (V_c/V_s) \]

Examples:

\[ V_c/V_s = 0.0001 \Rightarrow (V_c/V_s)_\text{dB} = -80 \text{ dB} \]

\[ V_c/V_s = 0.707 \Rightarrow (V_c/V_s)_\text{dB} = -3 \text{ dB} \]

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Finding the Phase

\[ j\omega v_c e^{i\phi} + e^{i\phi} \]

(a real number)

Use Euler’s formula to convert to rectangular form:

\[ j\omega t(\cos \phi + j\sin \phi) + (\cos \phi + j\sin \phi) = v_s / v_c \]

Collect real and imaginary parts; latter must be zero:

\[ \text{Im}(\cdot) = \omega t \cos \phi + \sin \phi = 0 \]

\[ \tan \phi = -\omega t \]
Graphical Result for Phase $\phi$

Finding the “Real” Waveform

- How to connect the imaginary exponential solution to the measured waveform $v(t)$? Conventionally, $v(t)$ is the real part of the of the imaginary exponential

$$\text{Re}(ve^{i(\omega t + \phi)}) = v \cos(\omega t + \phi)$$

Pushing This Idea Further …

There are two parameters needed to define a sinusoidal signal:
- magnitude
- phase

Why not work with a complex number as the signal and eliminate the imaginary exponential from the analysis (it cancelled out)?

Define the complex number consisting of the amplitude and phase a sinusoidal signal as a phasor

$$v(t) = v \cos(\omega t + \phi) \Leftrightarrow v(t) = Ve^{i\omega t}$$
$$V = ve^{i\phi}$$