Lecture 6

• Last time:
  – Rapid sketching techniques for more complicated transfer functions

• Today:
  – 2nd order circuits in the time and frequency domains

A Second Order System

Where does the inductor come from?

Do step response: \( v_s(t) \) jumps to \( V_{DD} \) at \( t = 0 \)
Step Response of L-R-C Circuit

Initial conditions: \( v_C(t=0) = 0 \text{ V}; \ i_L(t=0) = 0 \text{ A} \)

\[
i_L = i_C \quad \rightarrow \quad \left( \frac{1}{L} \right) \int_0^t v_L(t') dt' = C \frac{dv_C}{dt}
\]

Inductor voltage: \( v_L = V_{DD} - \left( \frac{C}{dt} \right) R + v_C \)

Solving the 2\text{nd} Order ODE

\[
LC \frac{d^2 v_C}{dt^2} + RC \frac{dv_C}{dt} + v_C = V_{DD}
\]

Steady-state solution: \( v_{C,ss} = V_{DD} \ (t \rightarrow \infty) \)

Transient solution: \( v_{C,tr} = ? \ldots \) guess \( v_{C,tr} = ae^{st} \)

and substitute: \( LC^2 \left( ae^{st} \right) + RC \left( ae^{st} \right) + ae^{st} = 0 \)

\[
s^2 + \left( R / L \right) s + \left( 1 / (LC) \right) = 0
\]
Characteristic Equation

\[ s^2 + \left( \frac{R}{L} \right)s + \left( \frac{1}{LC} \right) = 0 \]

Use quadratic formula to find the roots:

\[ s_{1,2} = -\left( \frac{R}{2L} \right) \pm \sqrt{\left( \frac{R}{2L} \right)^2 - \left( \frac{1}{LC} \right)} \]

Underdamped Case

\[ s_{1,2} = -\left( \frac{R}{2L} \right) \pm \sqrt{\left( \frac{R}{2L} \right)^2 - \left( \frac{1}{LC} \right)} = -\left( \frac{R}{2L} \right) \pm j\sqrt{\left( \frac{1}{LC} \right) - \left( \frac{R}{2L} \right)^2} \]

\[ v_C(t) = V_{DD} + a_1 e^{-(R/2L)t} e^{j\sqrt{(1/CL) - (R/2L)^2}t} + a_2 e^{-(R/2L)t} e^{-j\sqrt{(1/CL) - (R/2L)^2}t} \]

Form of solution …
Qualitative under-damped Waveform

Extreme under-damped Case

Exponential decay time is set by $\alpha = R/(2L)$

Small $R/L \to$ decay takes a long time and oscillation has a frequency that’s nearly $\sqrt{1/(LC)}$

Number of cycles during “ringdown” is

$$N \approx \frac{(1/\alpha)}{\left(1/\sqrt{1/(LC)}\right)} = \frac{2L/R}{\sqrt{LC}} = \frac{2}{R\sqrt{C}}$$

What happens when $R = 0 \, \Omega$?
thin-Film Bulk Acoustic Resonator (FBAR)

- Agilent Technologies
  IEEE ISSCC 2001
  2 GHz resonator
- $N > 1000$
- Brian Otis, Jan Rabaey (BWRC): low-noise oscillator
- Equivalent Circuit:

Phasor Analysis of 2nd Order Circuit

Impedance divider:

$$V_C = V_S \left[ \frac{1/ j\omega C}{(1/ j\omega C) + R + (j\omega L)} \right]$$
Transfer Function

Simplifying:

\[ H(j\omega) = \frac{1}{1 + j\omega RC - \omega^2 LC} \]

Define parameters: \( \omega_o = \frac{1}{\sqrt{LC}} \quad \tau = RC \)

\[ H(j\omega) = \frac{1}{1 - (\omega / \omega_o)^2 + j\omega \tau} \]

Limiting Cases: Magnitude and Phase

Low frequency: \( \omega << \omega_o \)

High frequency: \( \omega >> \omega_o \)

Resonant frequency: \( \omega = \omega_o \)
Inductor-Capacitor “Tuning”

At resonance, the impedance of the capacitor cancels the impedance of the inductor → phasor current is maximum and capacitor voltage peaks

How “sharp” or “narrow” is the resonance?

Define the quality factor \( Q = \frac{\omega_0}{\Delta \omega} \rightarrow Q = \frac{1}{\omega_0 \tau} \)

Magnitude Bode Plot
Phase Bode Plot

\[ \omega_0, \Delta \omega, \omega_0 + \Delta \omega \]