Lecture 6

• Last time:
  – Rapid sketching techniques for more complicated transfer functions

• Today: \(-R\text{L}C-\)
  \[\{\text{2}\text{nd order circuits in the time and frequency domains}\}\]
A Second Order System

- Where does the inductor come from?
- Do step response: $v_S(t)$ jumps to $V_{DD}$ at $t = 0$
Step Response of L-R-C Circuit

Initial conditions: \( v_C(t=0) = 0 \text{ V}; \ i_L(t=0) = 0 \text{ A} \)

\[
\begin{align*}
\text{Inductor voltage:} & \quad v_L = V_{DD} - \left( C \frac{dv_C}{dt} \right)R + v_C \\
& = \left( \int_0^t v_L(t')dt' \right)C \frac{dv_C}{dt}
\end{align*}
\]
Solving the 2nd Order ODE

\[ \frac{d^2 v_C}{dt^2} + \frac{RC}{LC} \frac{dv_C}{dt} + v_C = V_{DD} \]

Steady-state solution: \( v_{C,ss} = V_{DD} \) \((t \to \infty)\)

Transient solution: \( v_{C,ir} = a e^{st} \)

Guess \( v_{C,ir} = a e^{st} \)

\[ s^2 + \left( \frac{L}{R} \right)s + \left( \frac{1}{(LC)} \right) = 0 \]

and substitute: \( L \left( \frac{d^2}{dt^2} \right) + \frac{1}{C} \left( \frac{dv_C}{dt} \right) + \frac{R}{C} v_C = 0 \)
Characteristic Equation

Use quadratic formula to find the roots:

\[ s_1,2 = \frac{-R \pm \sqrt{(\frac{R}{2L})^2 - \frac{1}{LC}}}{2L} \]

\[ s_1,2 \approx \frac{-R \pm \sqrt{(\frac{R}{2L})^2 - \frac{1}{LC}}}{2L} \]

\[ \text{OVER Damped} \]

\[ \text{CRITICALLY DAMPED} \]

\[ \text{IDENTICAL ROOTS} \]

\[ \text{REAL ROOTS} \]

\[ \text{UN Under DAMPED} \]

\[ s \approx \text{COMPLEX} \]
Underdamped Case

\[ s_{1,2} = \left( \frac{R}{2L} \right) \pm \sqrt{\left( \frac{R}{2L} \right)^2 - \left( \frac{1}{LC} \right)} = \left( \frac{R}{2L} \right) \pm j\sqrt{\left( \frac{1}{LC} \right) - \left( \frac{R}{2L} \right)^2} \]

\[ \alpha = \sqrt{\frac{1}{LC} - \left( \frac{R}{2L} \right)^2} \]

\[ v_C(t) = V_{DD} + a_1 e^{-\left( \frac{R}{2L} \right)t} e^{j\sqrt{\left( \frac{1}{LC} \right) - \left( \frac{R}{2L} \right)^2} t} + \alpha e^{-\left( \frac{R}{2L} \right)t} e^{-j\sqrt{\left( \frac{1}{LC} \right) - \left( \frac{R}{2L} \right)^2} t} \]

Form of solution ...

\[ v_C(t) = V_{DD} + \left[ \cdots \right] e^{-\alpha t} \cos \left[ \omega_d t + \phi \right] \]
Qualitative Underdamped Waveform

$V_{pp} = 2.5V$

$\frac{e^{-at}}{\sqrt{1 - \frac{a^2}{\omega_0^2}}}$

$v(t) (V)$

$t (ns)$
Extreme Underdamped Case

Exponential decay time is set by $\alpha = \frac{R}{2L}$

Small $\frac{R}{L} \to$ decay takes a long time and oscillation has a frequency that's nearly $\sqrt{1/(LC)}$

Number of cycles during "ringdown" is

$N \approx \frac{\left(\frac{1}{\alpha}\right)}{\left(1/\sqrt{1/(LC)}\right)} = \frac{2L/R}{\sqrt{LC}} = \frac{2}{R\sqrt{C}}$

What happens when $R = 0$ Ω?

$\omega_d \rightarrow \omega_0 = \frac{1}{\sqrt{LC}}$
thin-Film Bulk Acoustic Resonator (FBAR)

Drive Electrode

Thin Piezoelectric Film

Sense Electrode

Agilent Technologies
IEEE ISSCC 2001

2 GHz resonator

N > 1000

Brian Otis, Jan Rabaey
(BWRC): low-noise oscillator

Equivalent Circuit:

R x 0.1
Phasor Analysis of 2nd Order Circuit

![Circuit Diagram]

\[ V_c = V_s \left[ \frac{1/j\omega C}{(1/j\omega C) + R + (j\omega L)} \right] \]

\[ f^2\omega^2 = L/C \]
Transfer Function

\[ H(j\omega) = \frac{1}{1 + j\omega RC - \omega^2 LC} \]

Define parameters:
\[ \omega_0 = \frac{1}{\sqrt{LC}} \]

Simplifying:
\[ H(j\omega) = \frac{1}{1 - (\omega/\omega_0)^2 + j\omega/\omega_0} \]

\[ \tau = RC \]

```
-NEW-

"STANDARD
FORM"
```
Limiting Cases: Magnitude and Phase

- Low frequency: $\omega \ll \omega_0$
  \[ |H|_{dB} = 0 \text{ dB} \quad (1) \]
  \[ \angle H = 0^\circ \]

- High frequency: $\omega \gg \omega_0$
  \[ |H|_{dB} = \left| \frac{1}{\left(1 - \frac{\omega^2}{\omega_0^2}ight)^2 + j\frac{\omega}{\omega_0}} \right| \]

Resonant frequency: $\omega = \omega_0$

\[ |H|_{dB} = \left| \frac{1}{\sqrt{1 - \frac{\omega^2}{\omega_0^2}}} \right| \]
\[ \angle H = -90^\circ \]

\[ \text{TRIG!} \]
\[ \angle H = \pm 180^\circ. \]
Inductor-Capacitor "Tuning"

At resonance, the impedance of the capacitor cancels the impedance of the inductor → phasor current is maximum and capacitor voltage peaks.

How "sharp" or "narrow" is the resonance?

Define the quality factor

\[ Q = \frac{\omega_0}{\Delta \omega} \]
\[ \omega = \omega_0 \quad \ldots \quad \frac{1}{j\omega} = \frac{1}{j\sqrt{\kappa c}} \cdot c = -j \frac{\sqrt{\kappa c}}{c} = \frac{1}{\sqrt{\kappa c}} = \frac{1}{\sqrt{\kappa c}} \]
Phase Bode Plot

\[ \Delta \omega = \frac{\omega_0}{Q} \]

\[ \omega_0 = 2\pi \left( 2 \times 10^9 \right) \text{rads/s} \]