Announcements

- Homework 8 due tomorrow 12 noon
- Lab 7 next week
- Reading: Chapter 10 (10.2, 10.3.2)
Lecture Material

- Last lecture
  - Bode plots
  - Second order functions
- This lecture
  - Finish second-order circuits
  - Frequency response of amplifiers

Low-Noise Amplifier

- D. Shaeffer, T. Lee, ISSCC’97

![Low-Noise Amplifier Diagram]
thin-Film Bulk Acoustic Resonator (FBAR)

- Agilent Technologies (IEEE ISSCC 2001)
- Q > 1000
- Resonates at 1.9 GHz
- Cell phone duplexer

Series LCR Step Response

- Consider the transient response of the following circuit when we apply a step at input
- Without inductor, the cap charges with RC time constant (EECS 40)
- Where does the inductor come from?
  - Intentional inductor placed in series
  - Every physical loop has inductance! (parasitic)
LCR Step Response: \( L \) Small

- We know the steady-state response is a constant voltage of \( V_{dd} \) across capacitor (inductor is short, cap is open)
- For the case of zero inductance, we know solution is of the following form:

\[
v_0(t) = V_{dd}(1 - e^{-t/\tau})
\]

LCR Circuit ODE

- Transient response solved in next few slides (A. Niknejad)
- Apply KVL to derive governing time-domain equations:

\[
v_s(t) = v_c(t) + v_r(t) + v_L(t)
\]

- Inductor and capacitor currents/voltages:

\[
i = i_c = C \frac{dv_c}{dt}, \quad v_L = L \frac{di}{dt}
\]

\[
v_L = L \frac{d}{dt} \left( C \frac{dv_c}{dt} \right) = LC \frac{d^2v_c}{dt^2}, \quad v_R = iR = RC \frac{dv_c}{dt}
\]

- We have the following 2\(^{nd}\) order ODE:

\[
v_s(t) = v_c(t) + RC \frac{dv_c}{dt} + LC \frac{d^2v_c}{dt^2}
\]
Initial Conditions

- For the solution of a second order circuit, we need to specify initial conditions (IC):
  \[ v_0(0) = v_C(0) = 0 \text{V} \]
  \[ i(0) = i_L(0) = 0 \text{V} \]

- For \( t > 0 \), the source voltage is \( V_{dd} \). Solve the following non-homogeneous equation subject to above IC:
  \[ V_{dd} = v_C(t) + RC \frac{dv_C}{dt} + LC \frac{d^2v_C}{dt^2} \]

- Steady state:
  \[ \frac{\text{d}}{\text{d}t} \rightarrow 0 \]
  \[ V_{dd} = v_C(\infty) \]

Guess Solution!

- Let's subtract out the steady-state solution:
  \[ v_C(t) = V_{dd} + v(t) \]
  \[ V_{dd} = \overrightarrow{V_{dd}} + \overrightarrow{v(t)} + RC \overrightarrow{\frac{dv}{dt}} + LC \overrightarrow{\frac{d^2v}{dt^2}} \]
  \[ 0 = \overrightarrow{v(t)} + RC \overrightarrow{\frac{dv}{dt}} + LC \overrightarrow{\frac{d^2v}{dt^2}} \]

- Guess solution is of the following form:
  \[ v(t) = Ae^{st} \]
  \[ 0 = Ae^{st} + RC \left( sAe^{st} \right) + LC \left( s^2 Ae^{st} \right) \]
  \[ 0 = Ae^{st} \left( 1 + RCs + LCs^2 \right) \]
  \[ 0 = 1 + RCs + LCs^2 \]
Again We're Back to Algebra

- Our guess is valid if we can find values of “s” that satisfy this equation:
  \[ 0 = 1 + RCS + LCs^2 \quad \rightarrow \quad 1 + (s \tau)2\zeta + (s \tau)^2 = 0 \]
  \[ \frac{Q}{2\zeta} = \frac{1}{\omega_0} \]

- The solutions are:
  \[ s \tau = -\zeta \pm \sqrt{\zeta^2 - 1} \]

- This is the same equation we solved in the last lecture!

- There we found three interesting cases:
  \[ \zeta = \begin{cases} 
  < 1 & \text{Underdamped} \\
  = 1 & \text{Critically damped} \\
  > 1 & \text{Overdamped} 
\end{cases} \]

General Case

- Solutions are real or complex conj depending on if \( \zeta > 1 \) or \( \zeta < 1 \)
  \[ s = \frac{1}{\tau}(-\zeta \pm \sqrt{\zeta^2 - 1}) = \begin{cases} 
  s_1 \\
  s_2 
\end{cases} \]

- \( v_C(t) = V_{dd} + A \exp(s_1 t) + B \exp(s_2 t) \)
- \( v_C(0) = V_{dd} + A + B = 0 \)

- \( i(0) = C \frac{dv_C(t)}{dt} \bigg|_{t=0} = 0 \quad \Rightarrow \quad As_1 \exp(s_1 t) + Bs_2 \exp(s_2 t) \bigg|_{t=0} = 0 \)
  \[ As_1 + Bs_2 = 0 \]
  \[ A + B = -V_{dd} \]
Final Solution (General Case)

Solve for $A$ and $B$:

$$A = \frac{-V_{dd}}{1 - \frac{s_1}{s_2}}$$

$$B = -V_{dd} - A = \frac{-V_{dd}(1 - \frac{s_1}{s_2}) + V_{dd}}{1 - \frac{s_1}{s_2}} = \frac{s_1 V_{dd}}{1 - \frac{s_1}{s_2}}$$

$$v_C(t) = V_{dd} + \frac{-V_{dd}}{1 - \frac{s_1}{s_2}} \exp(s_1 t) + \frac{s_1 V_{dd}}{1 - \frac{s_1}{s_2}} \exp(s_2 t)$$

$$v_C(t) = V_{dd} \left(1 - \frac{1}{1 - \frac{s_1}{s_2}} \left(e^{s_1 t} - \frac{s_1}{s_2} e^{s_2 t}\right)\right)$$

Overdamped Case

$\zeta > 1$: Time constants are real and negative

$$s = \frac{1}{\tau}(-\zeta \pm \sqrt{\zeta^2 - 1}) = \begin{cases} s_1 < 0 \\ s_2 < 0 \end{cases}$$

$\tau = 1$

$V_{dd} = 1$

$\zeta = 2$
Critically Damped

- $\zeta > 1$: Time constants are real and equal
  
  $$s = \frac{1}{\tau}(-\zeta \pm \sqrt{\zeta^2 - 1}) = -\frac{1}{\tau}$$

  $$\lim_{\zeta \to 1} \nu_C(t) = V_{dd}(1 - e^{-t/\tau} - te^{-t/\tau})$$

  $\tau = 1$
  $V_{dd} = 1$
  $\zeta = 1$

Underdamped

- Now the $s$ values are complex conjugate

  $$s_1 = a + jb$$
  $$s_2 = a - jb$$

  $$\nu_C(t) = V_{dd} + A\exp((a + jb)t) + B\exp((a - jb)t)$$
  $$\nu_C(t) = V_{dd} + e^{at}(A\exp(jbt) + B\exp(-jbt))$$

  $$A^* = \frac{-V_{dd}}{1 - \frac{s_1}{s_2}} = \frac{V_{dd}}{\frac{s_1}{s_2} - 1} = \frac{S_1}{S_2} V_{dd} = B$$

  $$\nu_C(t) = V_{dd} + e^{at}(A\exp(jbt) + A^*\exp(-jbt))$$
Underdamped (cont)

So we have:

\[ v_C(t) = V_{dd} + e^{at} \left( A \exp(jbt) + A^* \exp(-jbt) \right) \]
\[ v_C(t) = V_{dd} + e^{at} 2 \Re\{A \exp(jbt)\} \]
\[ v_C(t) = V_{dd} + e^{at} |A| \cos(\omega t + \phi) \]

\[ |A| = \frac{V_{dd}}{1 + \frac{s_1}{s_2}} \quad \phi = \angle \left( -\frac{V_{dd}}{1 + \frac{s_1}{s_2}} \right) \]

Underdamped Peaking

For \( \zeta < 1 \), the step response overshoots:

\[ \tau = 1 \]
\[ V_{dd} = 1 \]
\[ \zeta = 0.5 \]
Extremely Underdamped

\[ \tau = 1 \]
\[ V_{dd} = 1 \]
\[ \zeta = 0.01 \]

Common Source Amplifier: \( A_i(j\omega) \)

DC Bias is problematic: what sets \( V_{GS} \)?
Common Source: Discrete Biasing

With ideal MOS

CS Short-Circuit Current Gain

Transfer function: \[ A_I(j\omega) = \frac{g_m \left(1 - j\omega C_{gd} / g_m\right)}{j\omega(C_{gs} + C_{gd})} \]
Magnitude Bode Plot

Transition frequency:
Current gain = 1

MOS Unity Gain Frequency

Since the zero occurs at a higher frequency than pole, assume it has negligible effect:

\[ A_T \approx \frac{g_m}{j\omega(C_{gt} + C_{gd})} = 1 \quad \rightarrow \quad \omega_T = \frac{g_m}{(C_{gt} + C_{gd})} \]

\[ \omega_T \approx \frac{g_m}{C_{gt}} = \frac{\mu C_{ox}}{2} \left( \frac{W}{L} \right) (V_{GS} - V_T) \]

Performance improves with \( L^2 \) for long channel devices!
For short channel devices the dependence is \( \sim L^1 \)

\[ \omega_T \approx \frac{3}{2} \frac{\mu(V_{GS} - V_T)}{L^3} \sim \frac{\mu}{L} \frac{V_{GS} - V_T}{L} = \frac{\mu E_{eff}}{L} = \frac{v}{L} = \frac{\tau}{L} \]