Announcements

- Homework 8 due next Tuesday
- Lab 6 this week
- Lab 7 next week
- Reading: Chapter 10 (10.1)
Lecture Material

- Last lecture
  - Frequency-domain analysis
  - Bode plots
- This lecture
  - More Bode plots
  - Second order functions

Power Flow

- The instantaneous power flow into any element is the product of the voltage and current: \( P(t) = i(t)v(t) \)
- For a periodic excitation, the average power is:
  \[
  P_{av} = \int_T i(\tau)v(\tau)d\tau
  \]
- In terms of sinusoids we have
  \[
  P_{av} = \int_T |\text{\textbf{V}|}\cos(\omega t + \phi_i)|\text{\textbf{V}|}\cos(\omega t + \phi_v)d\tau
  \]
  \[
  = |\text{\textbf{V}|}\int_T (\cos(\omega t \cos \phi_i - \sin(\omega t \sin \phi_i)) \cdot (\cos(\omega t \cos \phi_v - \sin(\omega t \sin \phi_v))d\tau
  \]
  \[
  = |\text{\textbf{V}|}\int_T d\tau \cos^2(\omega t \cos \phi_i \cos \phi_v + \sin^2(\sin \phi_i \sin \phi_v) + c \sin(\omega t \cos \omega t
  \]
  \[
  = |\text{\textbf{V}|}\int_T \text{\textbf{V}|}(\cos \phi_i \cos \phi_v + \sin \phi_i \sin \phi_v) = \frac{|\text{\textbf{V}|}}{2} \cos(\phi_i - \phi_v)
Power Flow with Phasors

\[ P_{av} = \frac{|V|}{2} \cos(\phi_i - \phi_v) \]

Power Factor

- Note that if \((\phi_i - \phi_v) = \frac{\pi}{2}\), then \(P_{av} = \frac{|V|}{2} \cos(\pi/2) = 0\)
- From the previous slide:

\[ P = \frac{|V|^2}{2} \cos(\phi_i - \phi_v) = \frac{1}{2} \text{Re}[V \cdot V^*] = \frac{1}{2} \text{Re}[I^* \cdot V] \]

More Power

- In terms of the circuit impedance we have:

\[ P = \frac{1}{2} \text{Re}[I \cdot V^*] = \frac{1}{2} \text{Re}[\frac{V}{Z} \cdot V^*] = \frac{|V|^2}{2} \text{Re}[Z^{-1}] \]

\[ = \frac{|V|^2}{2} \frac{\text{Re}[Z^*]}{|Z|^2} = \frac{|V|^2}{2|Z|^2} \text{Re}[Z^*] = \frac{|V|^2}{2|Z|^2} \text{Re}[Z] \]

- Check the result for a real impedance (resistor)
- Also, in terms of current:

\[ P = \frac{1}{2} \text{Re}[I^* \cdot V] = \frac{1}{2} \text{Re}[I^* \cdot I \cdot Z] = \frac{|I|^2}{2} \text{Re}[Z] \]
Second Order Circuits

- The series resonant circuit is one of the most important elementary circuits:

- The physics describes not only electrical LCR circuits, but also approximates mechanical resonance (mass-spring, pendulum, molecular resonance, microwave cavities, transmission lines, buildings, bridges, ...)

Series LCR Impedance

- With phasor analysis, this circuit is readily analyzed

\[
Z = j\omega L + \frac{1}{j\omega C} + R
\]

\[
Z = j\omega L + \frac{1}{j\omega C} + R = R + j\omega L \left(1 - \frac{1}{\omega^2 LC}\right)
\]

\[
\text{Im}[Z] = \omega L \left(1 - \frac{1}{\omega^2 LC}\right) = 0 \quad \omega^2 = \frac{1}{LC}
\]
Resonance

- Resonance occurs when the circuit impedance is purely real
- Imaginary components of impedance cancel out
- For a series resonant circuit, the current is maximum at resonance

\[ \text{Series Resonance Voltage Gain} \]

- Note that at resonance, the voltage across the inductor and capacitor can be larger than the input voltage:

\[ V_L = I_j \omega_0 L = \frac{V_S}{Z(\omega_0)} j \omega_0 L = \frac{V_S}{R} j \omega_0 L = jQ \times V_S \]

\[ V_C = I \frac{1}{j \omega_0 C} = \frac{V_S}{Z(\omega_0)} \frac{\omega_0 L}{j} = -\frac{V_S}{R} j \omega_0 L = -jQ \times V_S \]

\[ Q = \frac{\omega_0 L}{R} = \frac{1}{\omega_0 C} \frac{1}{R} = \frac{\sqrt{LC}}{R} \frac{1}{R} = \frac{L}{C} \frac{1}{R} = \frac{Z_0}{R} \]
Second Order Transfer Function

- So we have:

\[
H(j\omega) = \frac{V_0}{V_s} = \frac{R}{j\omega L + \frac{1}{j\omega C} + R}
\]

- To find the poles/zeros, let’s put the \( H \) in canonical form:

\[
H(j\omega) = \frac{V_0}{V_s} = \frac{j\omega CR}{1 - \omega^2 LC + j\omega RC}
\]

- One zero at DC frequency \( \Rightarrow \) no DC current through a capacitor

Poles of 2\textsuperscript{nd} Order Transfer Function

- Denominator is a quadratic polynomial:

\[
H(j\omega) = \frac{V_0}{V_s} = \frac{j\omega CR}{1 - \omega^2 LC + j\omega RC} = \frac{j\omega R}{L} \left( \frac{1}{LC} + (j\omega)^2 + j\omega R \right)
\]

\[
H(j\omega) = \frac{j\omega R}{\omega_0^2 + (j\omega)^2 + j\omega R} \quad \omega_0^2 \equiv \frac{1}{LC}
\]

\[
H(j\omega) = \frac{j\omega_0^2}{\omega_0^2 + (j\omega)^2 + j\omega_0 R} \quad Q \equiv \frac{\omega_0 L}{R}
\]
Finding the poles...

- Let's factor the denominator:
  \[(j\omega)^2 + j\frac{\omega\omega_0}{Q} + \omega_0^2 = 0\]
  \[\omega = -\frac{\omega_0}{2Q} \pm \sqrt{\frac{\omega_0^2}{4Q^2} - \omega_0^2} = -\frac{\omega_0}{2Q} \pm j\omega_0 \sqrt{\frac{1}{4Q^2} - 1}\]

- Poles are complex conjugate frequencies
- The Q parameter is called the "quality-factor" or Q-factor
- This is an important parameter:
  \[Q \rightarrow R \rightarrow \infty\]

Resonance without Loss

- The transfer function can parameterized in terms of loss. First, take the lossless case, \(R=0\):
  \[\omega = -\frac{\omega_0}{2Q} \pm \sqrt{\frac{\omega_0^2}{4Q^2} - \omega_0^2} \rightarrow \pm j\omega_0\]

- When the circuit is lossless, the poles are at real frequencies, so the transfer function blows up!
- At this resonance frequency, the circuit has zero imaginary impedance and thus zero total impedance
- Even if we set the source equal to zero, the circuit can have a steady-state response (oscillates)
Magnitude Response

- The response peakiness depends on $Q$

$$H(j\omega) = \frac{j\omega_0 \frac{R}{L}}{\omega_0^2 - \omega^2 + j\omega \frac{\omega_0 R}{L}} = \frac{j\omega \omega_0}{Q^{\alpha}}$$

How Peaky is it?

- Let's find the points when the transfer function squared has dropped in half:

$$|H(j\omega)|^2 = \frac{\left(\frac{\omega_0}{Q}\right)^2}{\left(\omega_0^2 - \omega^2\right)^2 + \left(\frac{\omega_0}{Q}\right)^2} = \frac{1}{2}$$

$$|H(j\omega)|^2 = \frac{1}{\left(\frac{\omega_0^2 - \omega^2}{\omega_0 \omega/Q} + 1\right) + 1} = \frac{1}{2}$$

$$\left(\frac{\omega_0^2 - \omega^2}{\omega_0 \omega/Q}\right) = 1$$
Half Power Frequencies (Bandwidth)

- We have the following:
  \[
  \left(\frac{\omega_0^2 - \omega^2}{\omega_0/\omega}\right)^2 = 1 \quad \rightarrow \quad \frac{\omega_0^2 - \omega^2}{\omega_0/\omega} = \pm 1
  \]
  \[
  \omega^2 \mp \frac{\omega_0^2}{Q} - \omega_0^2 = 0
  \]
  \[
  \omega = \pm \frac{\omega_0}{2Q} \pm \sqrt{\left(\frac{\omega_0}{Q}\right)^2 + \omega_0^2} = \pm a \pm b \quad b > a \quad \rightarrow \quad + a + b > 0 \quad + a - b < 0 \quad - a + b > 0 \quad - a - b < 0
  \]
  Take positive frequencies:
  \[
  \Delta \omega = \omega_+ - \omega_- = \frac{\omega_0}{Q} \quad \frac{\Delta \omega}{\omega_0} = \frac{1}{Q}
  \]

More “Notation”

- Often a second-order transfer function is characterized by the “damping” factor as opposed to the “Quality” factor
  \[
  \omega_0^2 + (j\omega)^2 + \frac{j\omega\omega_0}{Q} = 0 \quad \tau = \frac{1}{\omega_0}
  \]
  \[
  1 + (j\omega\tau)^2 + \frac{j\omega\tau}{Q} = 0
  \]
  \[
  1 + (j\omega\tau)^2 + (j\omega\tau)2\zeta = 0
  \]
  \[
  Q = \frac{1}{2\zeta}
  \]
Second Order Circuit Bode Plot

- Quadratic poles or zeros have the following form:
  \[(j\omega\tau)^2 + (j\omega\tau)2\zeta + 1 = 0\]
  damping ratio

- The roots can be parameterized in terms of the damping ratio:
  \[
  \zeta = 1 \quad \Rightarrow \quad (j\omega\tau)^2 + (j\omega\tau)2 + 1 = (1 + j\omega\tau)^2
  
  Two equal poles
  
  \[
  \zeta > 1 \quad \Rightarrow \quad (j\omega\tau)^2 + (j\omega\tau)2\zeta + 1 = (1 + j\omega\tau_1)(1 + j\omega\tau_2)
  
  Two real poles
  
  \[j\omega\tau = -\zeta \pm \sqrt{\zeta^2 - 1}\]

Bode Plot: Damped Case

- The case of \(\zeta > 1\) and \(\zeta = 1\) is a simple generalization of simple poles (zeros). In the case that \(\zeta > 1\), the poles (zeros) are at distinct frequencies. For \(\zeta = 1\), the poles are at the same real frequency:

  \[
  \zeta = 1 \quad \Rightarrow \quad (j\omega\tau)^2 + (j\omega\tau)2 + 1 = (1 + j\omega\tau)^2
  
  Asymptotic Slope is 40 dB/dec
  
  \[|1 + j\omega\tau|^2 = |1 + j\omega\tau|^2\]

  \[20\log|1 + j\omega\tau|^2 = 40\log|1 + j\omega\tau|\]

  Asymptotic Phase Shift is 180°
Underdamped Case

- For $\zeta < 1$, the poles are complex conjugates:
  \[
  (j\omega\tau)^2 + (j\omega\tau)2\zeta + 1 = 0
  \]
  
  \[j\omega\tau = -\zeta \pm \sqrt{\zeta^2 - 1} = \zeta \pm j\sqrt{1 - \zeta^2}
  \]

- For $\omega\tau \ll 1$, this quadratic is negligible (0dB)
- For $\omega\tau \gg 1$, we can simplify:
  \[
  20\log\left|(j\omega\tau)^2 + (j\omega\tau)2\zeta + 1\right| \approx 20\log|(j\omega\tau)^2| = 40\log|\omega\tau|
  \]

- In the transition region $\omega\tau \sim 1$, things are tricky!

Underdamped Mag Plot
Underdamped Phase

- The phase for the quadratic factor is given by:
  \[ \angle((j\omega \tau)^2 + (j\omega \tau)2\zeta + 1) = \tan^{-1}\left(\frac{2\omega \tau \zeta}{1 - (\omega \tau)^2}\right) \]

- For \( \omega \tau < 1 \), the phase shift is less than 90°
- For \( \omega \tau = 1 \), the phase shift is exactly 90°
- For \( \omega \tau > 1 \), the argument is negative so the phase shift is above 90° and approaches 180°
- Key point: argument shifts sign around resonance

Phase Bode Plot
Bode Plot Guidelines

- In the transition region, note that at the breakpoint:
  \[(j\omega \tau)^2 + (j\omega \tau)2\zeta + 1 = (j)^2 + (j)2\zeta + 1 = 2\zeta = \frac{1}{Q}\]

- From this you can estimate the peakiness in the magnitude response.

- Example: For \(\zeta = 0.1\), the Bode magnitude plot peaks by 20 log(5) \(\sim 14\) dB

- The phase is much more difficult. Note for \(\zeta = 0\), the phase response is a step function

- For \(\zeta = 1\), the phase is two real poles at a fixed frequency

- For \(0 < \zeta < 1\), the plot should go somewhere in between!