Lecture 5: 2nd Order Circuits in the Time Domain
Physics of Conduction

Prof. Niknejad
Lecture Outline

- Second order circuits:
  - Time Domain Response
- Physics of Conduction
Series LCR Step Response

- Consider the transient response of the following circuit when we apply a step at input.
- Without inductor, the cap charges with RC time constant (EECS 40).
- Where does the inductor come from?
  - Intentional inductor placed in series
  - Every physical loop has inductance! (parasitic)
LCR Step Response: $L$ Small

- We know the steady-state response is a constant voltage of $V_{dd}$ across capacitor (inductor is short, cap is open)

- For the case of zero inductance, we know solution is of the following form:

\[ v_0(t) = V_{dd} \left(1 - e^{-t/\tau} \right) \]
**LCR Circuit ODE**

- Apply KVL to derive governing dynamic equations:
  \[ v_s(t) = v_C(t) + v_R(t) + v_L(t) \]

- Inductor and capacitor currents/voltages take the form:
  \[ i = i_C = C \frac{dv_C}{dt} \quad \nu_L = L \frac{di}{dt} \]
  \[ \nu_L = L \frac{d}{dt} \left( C \frac{dv_C}{dt} \right) = LC \frac{d^2v_C}{dt^2} \]
  \[ \nu_R = iR = RC \frac{dv_C}{dt} \]

- We have the following 2\textsuperscript{nd} order ODE:
  \[ \nu_s(t) = \nu_C(t) + RC \frac{dv_C}{dt} + LC \frac{d^2v_C}{dt^2} \]
Initial Conditions

- For the solution of a second order circuit, we need to specify initial conditions (IC):
  \[ v_0(0) = v_C(0) = 0 \text{ V} \]
  \[ i(0) = i_L(0) = 0 \text{ V} \]

- For \( t > 0 \), the source voltage is \( V_{dd} \). We can now solve for the following non-homogeneous equation subject to the above IC:
  \[ V_{dd} = v_C(t) + RC \frac{dv_C}{dt} + LC \frac{d^2v_C}{dt^2} \]

- Steady state: \( \frac{d}{dt} \to 0 \)
  \[ V_{dd} = v_C(\infty) \]
Guess Solution!

- Let’s subtract out the steady-state solution:

\[ v_c(t) = V_{dd} + v(t) \]

\[ V_{dd} = V_{dd} + v(t) + RC \frac{dv}{dt} + LC \frac{d^2v}{dt^2} \]

\[ 0 = v(t) + RC \frac{dv}{dt} + LC \frac{d^2v}{dt^2} \]

- Guess solution is of the following form:

\[ v(t) = Ae^{st} \]

\[ 0 = Ae^{st} + RC(sAe^{st}) + LC(s^2 Ae^{st}) \]

\[ 0 = Ae^{st}(1 + RCs + LCs^2) \]

\[ 0 = 1 + RCs + LCs^2 \]
Again We’re Back to Algebra

- Our guess is valid if we can find values of “s” that satisfy this equation:

\[
0 = 1 + RCs + LCs^2 \quad \rightarrow \quad 1 + (s\tau)2\zeta + (s\tau)^2 = 0
\]

- The solutions are:

\[
s\tau = -\zeta \pm \sqrt{\zeta^2 - 1}
\]

- This is the same equation we solved last lecture!

- There we found three interesting cases:

\[
\zeta = \begin{cases} 
< 1 & \text{Underdamped} \\
1 & \text{Critically Damped} \\
> 1 & \text{Overdamped}
\end{cases}
\]
General Case

- Solutions are real or complex conj depending on if \( \zeta > 1 \) or \( \zeta < 1 \)

\[
s = \frac{1}{\tau} (-\zeta \pm \sqrt{\zeta^2 - 1}) = \begin{cases} s_1 \\ s_2 \end{cases}
\]

\[
\nu_C(t) = V_{dd} + A \exp(s_1 t) + B \exp(s_2 t)
\]

\[
\nu_C(0) = V_{dd} + A + B = 0
\]

\[
i(0) = C \frac{d\nu_C(t)}{dt} \bigg|_{t=0} = 0 \quad \Rightarrow \quad A s_1 \exp(s_1 t) + B s_2 \exp(s_2 t) \bigg|_{t=0} = 0
\]

\[
A s_1 + B s_2 = 0
\]

\[
A + B = -V_{dd}
\]
Final Solution (General Case)

- Solve for $A$ and $B$:

\[
A = \frac{-V_{dd}}{1 - \frac{s_1}{s_2}} \\
B = -V_{dd} - A = \frac{-V_{dd} \left(1 - \frac{s_1}{s_2}\right) + V_{dd}}{1 - \frac{s_1}{s_2}} = \frac{s_1 V_{dd}}{s_2} \frac{s_1 V_{dd}}{1 - \frac{s_1}{s_2}}
\]

\[
v_C(t) = V_{dd} + \frac{-V_{dd}}{1 - \frac{s_1}{s_2}} \exp(s_1 t) + \frac{s_1 V_{dd}}{s_2} \frac{s_1 V_{dd}}{1 - \frac{s_1}{s_2}} \exp(s_2 t)
\]

\[
v_C(t) = V_{dd} \left[1 - \frac{1}{1 - \frac{s_1}{s_2}} \left(e^{s_1 t} - \frac{s_1}{s_2} e^{s_2 t}\right)\right]
\]
Overdamped Case

- \( \zeta > 1 \): Time constants are real and negative

\[
s = \frac{1}{\tau} (-\zeta \pm \sqrt{\zeta^2 - 1}) = \begin{cases} s_1 < 0 \\ s_2 \end{cases}
\]

\( \tau = 1 \)
\( V_{dd} = 1 \)
\( \zeta = 2 \)
Critically Damped

- $\zeta > 1$: Time constants are real and equal

\[
s = \frac{1}{\tau} (-\zeta \pm \sqrt{\zeta^2 - 1}) = -\frac{1}{\tau}
\]

\[
\lim_{\zeta \to 1} v_C(t) = V_{dd} \left(1 - e^{-t/\tau} - te^{-t/\tau}\right)
\]

\[
\tau = 1
\]

\[
V_{dd} = 1
\]

\[
\zeta = 1
\]
Now the $s$ values are complex conjugate

\[ s_1 = a + jb \]
\[ s_2 = a - jb \]

\[ v_c(t) = V_{dd} + A \exp((a + jb)t) + B \exp((a - jb)t) \]
\[ v_c(t) = V_{dd} + e^{at} \left( A \exp(jbt) + B \exp(-jbt) \right) \]

\[
A^* = \frac{-V_{dd}}{1 - \frac{s_1}{s_2}} = \frac{V_{dd}}{s_2 - 1} = \frac{s_1 V_{dd}}{1 - \frac{s_1}{s_2}} = B
\]

\[ v_c(t) = V_{dd} + e^{at} \left( A \exp(jbt) + A^* \exp(-jbt) \right) \]
Underdamped (cont)

- So we have:

\[ v_c(t) = V_{dd} + e^{at} \left( A \exp(jbt) + A^* \exp(-jbt) \right) \]

\[ v_c(t) = V_{dd} + e^{at} 2 \text{Re}[A \exp(jbt)] \]

\[ v_c(t) = V_{dd} + e^{at} 2|A| \cos(\omega t + \phi) \]

\[ |A| = \frac{V_{dd}}{1 + \frac{s_1}{s_2}} \]

\[ \phi = \angle\frac{V_{dd}}{1 + \frac{s_1}{s_2}} \]
Underdamped Peaking

- For $\zeta < 1$, the step response overshoots:

\[
\tau = 1 \\
V_{dd} = 1 \\
\zeta = .5
\]
Extremely Underdamped

\[ \tau = 1 \]
\[ V_{dd} = 1 \]
\[ \zeta = 0.01 \]
Ohm’s Law

- One of the first things we learn as EECS majors is:

\[ V = I \times R \]

- Is this trivial? Maybe what’s really going on is the following:

\[ V = f(I) = f(0) + f'(0)I + f''(0)I^2 / 2 + \ldots \approx f'(0)I \]

- In the above Taylor expansion, if the voltage is zero for zero current, then this is generally valid

- The range of validity (radius of convergence) is the important question. It turns out to be VERY large!
Ohm’s Law Revisited

- In Physics we learned:

\[ J = \sigma E \]

- Is this also trivial? Well, it’s the same as Ohm’s law, so the questions are related. For a rectangular solid:

\[ J = \frac{I}{A} = \sigma \frac{V}{L} \quad V = \frac{L}{\sigma A} I = RI \]

- Isn’t it strange that current (velocity) is proportional to Force?
- Where does conductivity come from?
Conductivity of a Gas

- Electrical conduction is due to the motion of positive and negative charges
- For water with pH=7, the concentration of hydrogen H$^+$ ions (and OH$^-$) is:

$$10^{-7} \text{ mole/L} = 10^{-10} \text{ mole/cm}^3 = 10^{-10} \times 6.02 \times 10^{23} \text{ cm}^{-3} = 6 \times 10^{13} \text{ cm}^{-3}$$

- Typically, the concentration of charged carriers is much smaller than the concentration of neutral molecules
- The motion of the charged carriers (electrons, ions, molecules) gives rise to electrical conduction
Collisions in Gas

- At a temperate $T$, each charged carrier will move in a random direction and velocity until it encounters a neutral molecule or another charged carrier.
- Since the concentration of charged carriers is much less than molecules, it will most likely encounter a molecule.
- For a gas, the molecules are widely separated (~10 molecular diameters).
- After colliding with the molecule, there is some energy exchange and the charge carrier will come out with a new velocity and new direction.
Memory Loss in Collisions

- Schematically

- Key Point: The initial velocity and direction is lost (randomized) after a few collisions
Application of Field

- When we apply an electric field, during each “free flight”, the carriers will gain a momentum of $E qt$
- Therefore, after $t$ seconds, the momentum is given by:

$$Mu + Eqt$$

- If we take the average momentum of all particles at any given time, we have:

$$M\bar{u} = \frac{1}{N} \sum_{j} (Mu_{j} + Eqt_{j})$$
Random Things Sum to Zero!

- When we sum over all the random velocities of the particles, we are averaging over a large number of random variables with zero mean, the average is zero

\[
M\overline{\mathbf{u}} = \frac{1}{N} \sum_j (M\mathbf{u}_j + Eqt_j)
\]

\[
M\overline{\mathbf{u}} = \frac{1}{N} \sum_j Eqt_j = Eqt
\]

\[
J = Nq\overline{\mathbf{u}} = Nq \left( \frac{Eqt}{M} \right) = Nq^2 \frac{\tau}{M} E = \sigma E
\]
Negative and Positive Carriers

Since current is contributed by positive and negative charge carriers:

\[ J = J^+ - J^- = e \left( \frac{N^+ e \tau^+}{M^+} - \frac{-N^- e \tau^-}{M^-} \right) E \]

\[ \sigma = e^2 \left( \frac{N^+ \tau^+}{M^+} + \frac{N^- \tau^-}{M^-} \right) \]
Conduction in Metals

- High conductivity of metals is due to large concentration of free electrons.
- These electrons are not attached to the solid but are free to move about the solid.
- In metal sodium, each atom contributes a free electron: \( N = 2.5 \times 10^{22} \text{ atoms/cm}^3 \)
- From the measured value of conductivity (easy to do), we can back calculate the mean free time:

\[
\tau = \frac{\sigma m}{Ne^2} = \frac{(1.9 \times 10^{17})(9 \times 10^{-28})}{(2.5 \times 10^{22})(23 \times 10^{-20})} = 3 \times 10^{-14} \text{ sec}
\]
A Deep Puzzle

- This value of mean free time is surprisingly long
- The mean velocity for an electron at room temperature is about:
  \[ \frac{mv^2}{2} = \frac{3}{2} kT \quad v = 3 \times 10^7 \text{ cm/sec} \]
- At this speed, the electron travels \( v\tau = 3 \times 10^{-7} \text{ cm} \)
- The molecular spacing between adjacent ions is only \( 3.8 \times 10^{-8} \text{ cm} \)
- Why is it that the electron is on average zooming by 10 positively charged ions?
The free carrier can penetrate right through positively charged host atoms!

Quantum mechanics explains this! (Take Physics 117A/B)

For a periodic arrangement of potential functions, the electron does not scatter. The influence of the crystal is that it will travel freely with an effective mass.

So why does it scatter at all?
Scattering in Metals

- At temperature $T$, the atoms are in random motion and thus upset periodicity

- Even at extremely low temperatures, the presence of an impurity upsets periodicity
Summary of Conduction

\[ \sigma = e^2 \left( \frac{N^+ \tau^+}{M^+} + \frac{N^- \tau^-}{M^-} \right) \]

- Conductivity determined by:
- Density of free charge carriers (both positive and negative)
- Charge of carrier (usually just \(e\))
- Effective mass of carrier
- Mean relaxation time (time for memory loss … usually the time between collisions)
  - This is determined by several mechanisms, e.g.:
    - Scattering by impurities
    - Scattering due to vibrations in crystal
Reference