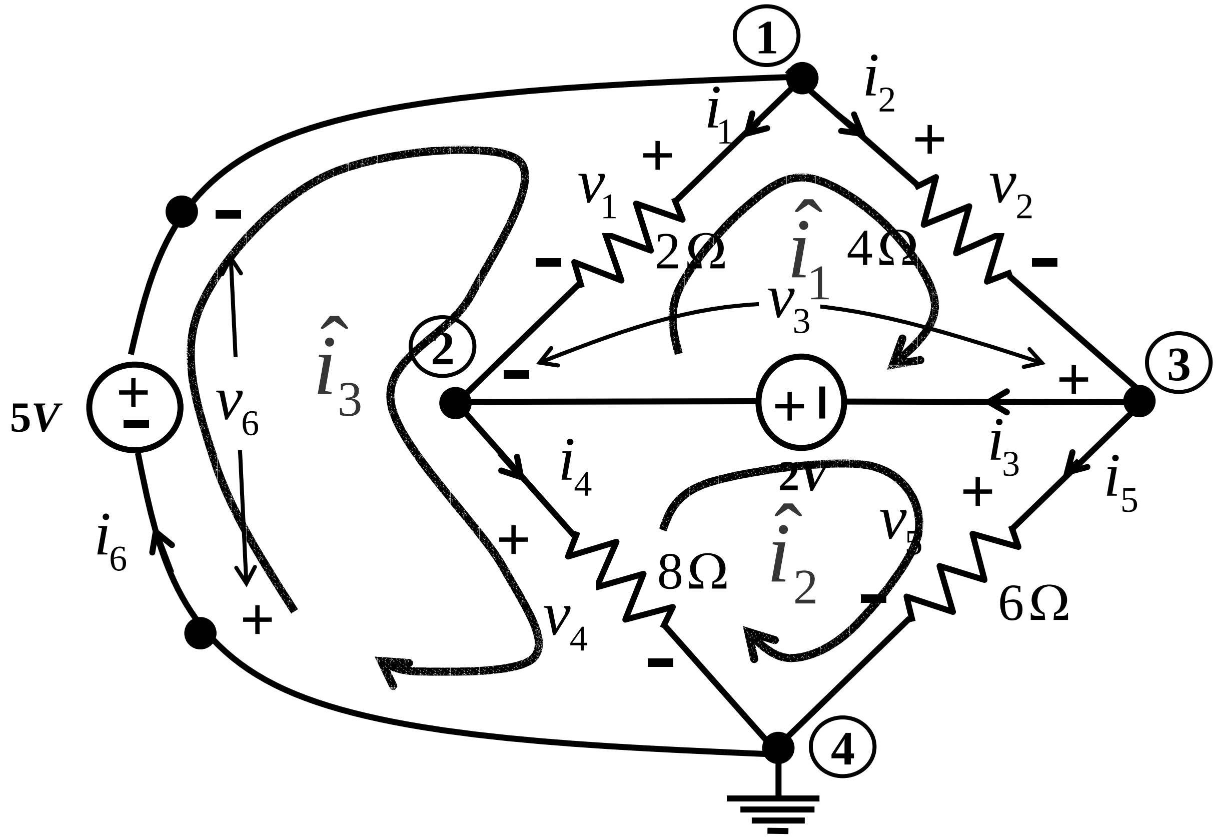


Mesh 1: $6\hat{i}_1 - 2\hat{i}_3 = 2$ (1)

Mesh 2: $14\hat{i}_2 - 8\hat{i}_3 = -2$ (2)

Mesh 3: $-2\hat{i}_1 - 8\hat{i}_2 + 10\hat{i}_3 = 5$ (3)

Solving $\hat{i}_1$ from (1) $\Rightarrow$ $\hat{i}_1 = \frac{1}{3}\hat{i}_3 - \frac{2}{3}$ (4)

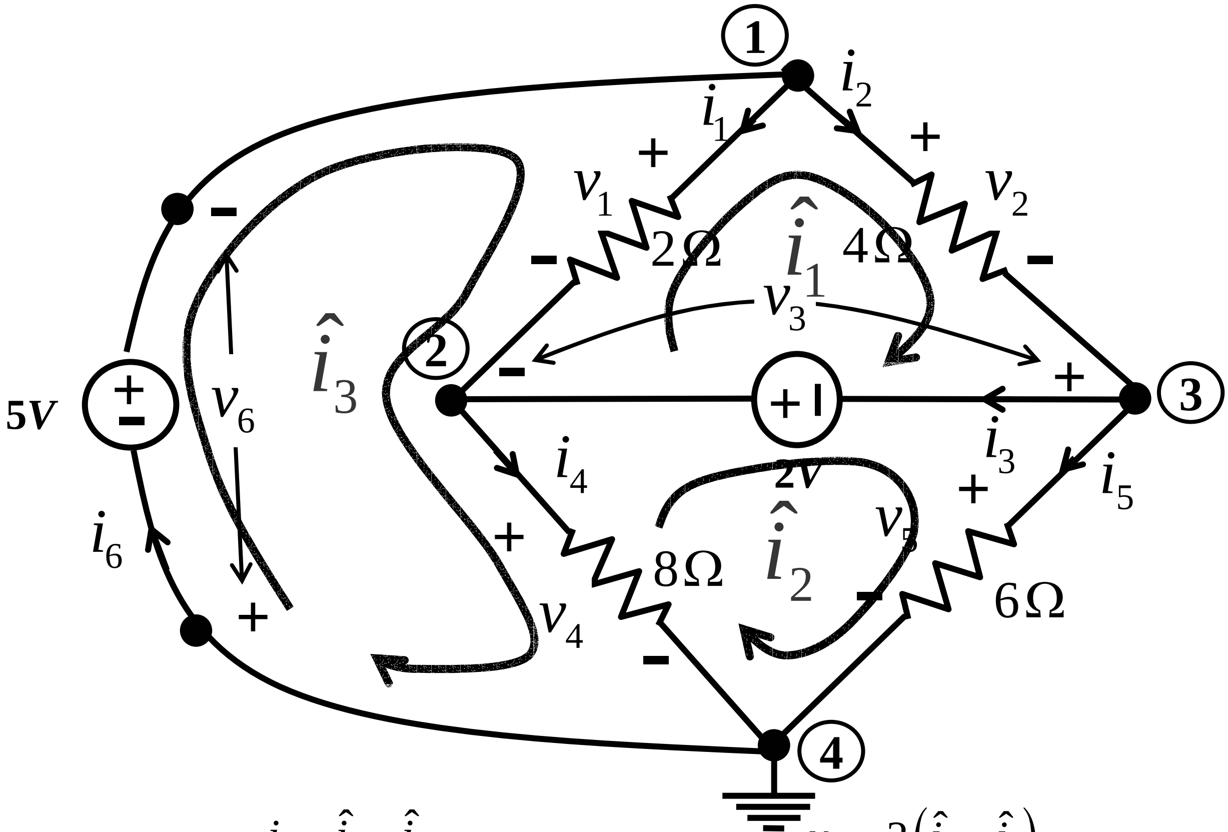
Solving $\hat{i}_2$ from (2) $\Rightarrow$ $\hat{i}_2 = \frac{4}{7}\hat{i}_3 - \frac{1}{7}$ (5)

Substituting (4) and (5) into (3) $\Rightarrow$

$$\hat{i}_3 = \frac{19}{12} A \quad (6)$$

(5) and (6) $\Rightarrow$ $\hat{i}_2 = \frac{8}{20} A$ (7)

(4) and (6) $\Rightarrow$ $\hat{i}_1 = \frac{13}{20} A$ (8)



$$i_1 = \hat{i}_3 - \hat{i}_1 \quad , \quad v_1 = 2\left(\hat{i}_3 - \hat{i}_1\right)$$
$$i_2 = \hat{i}_1 \quad , \quad v_2 = 4\hat{i}_1$$
$$i_3 = \hat{i}_1 - \hat{i}_2 \quad , \quad v_3 = -2$$
$$i_4 = \hat{i}_3 - \hat{i}_2 \quad , \quad v_4 = 8\left(\hat{i}_3 - \hat{i}_2\right)$$
$$i_5 = \hat{i}_2 \quad , \quad v_5 = 6\hat{i}_2$$
$$i_6 = \hat{i}_3 \quad , \quad v_6 = -5$$

Loop equation around mesh 1:

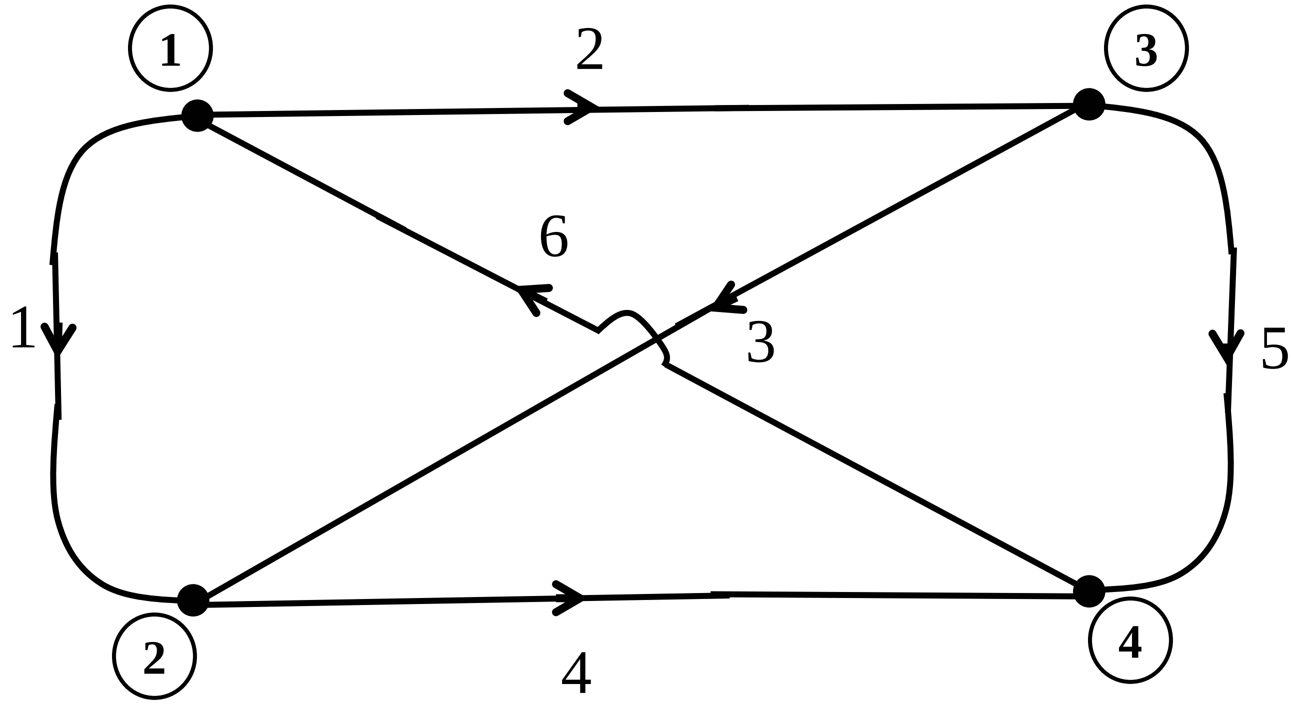
$$-v_1 + v_2 + v_3 = 0 \quad \Rightarrow \quad -2(\hat{i}_3 - \hat{i}_1) + 4\hat{i}_1 - 2 = 0$$
$$\Rightarrow \quad 6\hat{i}_1 - 2\hat{i}_3 = 2 \qquad (1)$$

Loop equation around mesh 2:

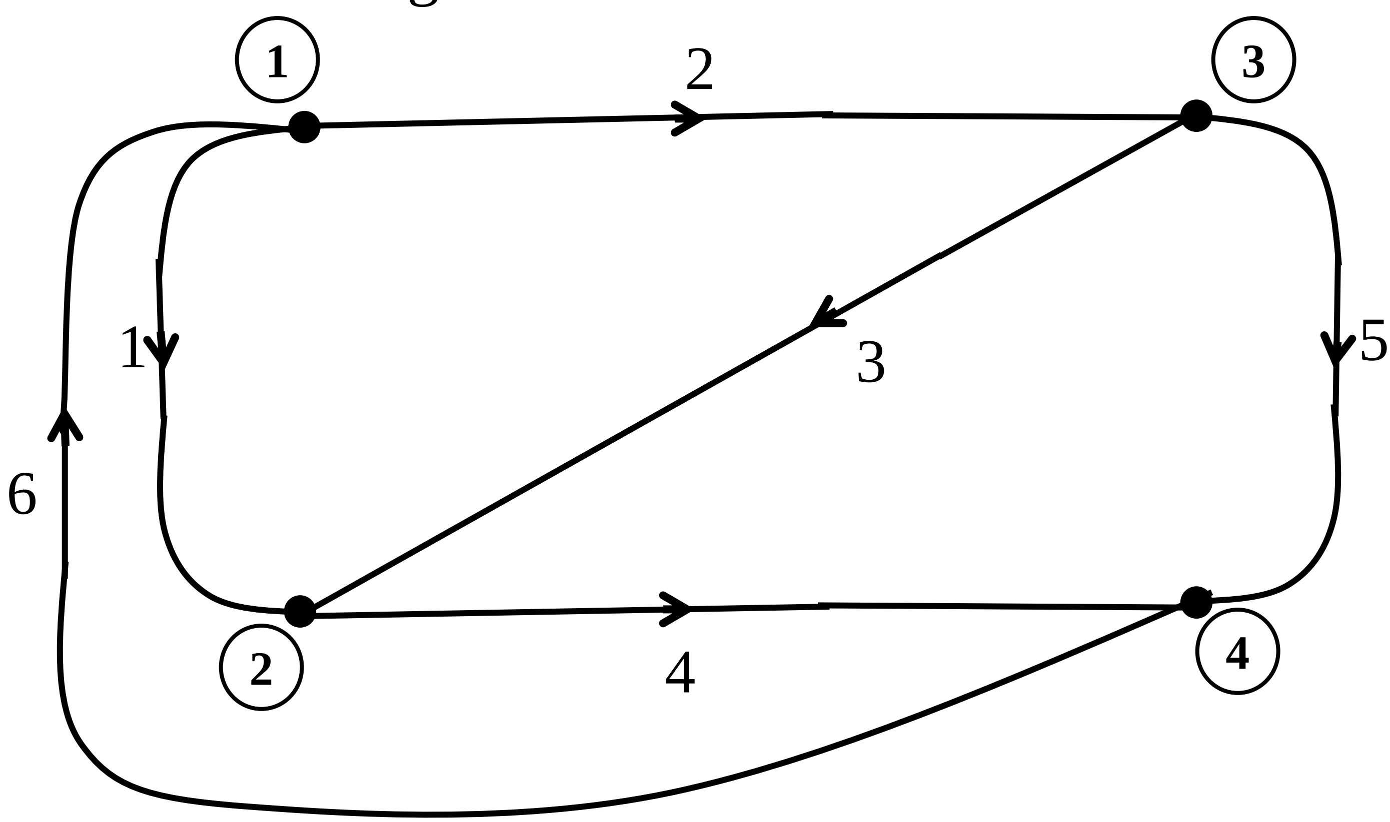
$$-v_3 + v_5 - v_4 = 0 \quad \Rightarrow \quad -(-2) + 6\hat{i}_2 - 8(\hat{i}_3 - \hat{i}_2) = 0$$
$$\Rightarrow \quad 14\hat{i}_2 - 8\hat{i}_3 = -2 \qquad (2)$$

Loop equation around mesh 3:

$$v_6 + v_1 + v_4 = 0 \quad \Rightarrow \quad -5 + 2(\hat{i}_3 - \hat{i}_1) + 8(\hat{i}_3 - \hat{i}_2) = 0$$
$$\Rightarrow \quad -2\hat{i}_1 - 8\hat{i}_2 + 10\hat{i}_3 = 5 \qquad (3)$$



We can redraw this digraph so that there are no intersecting branches.



Hence the above digraph is **planar**.

Writing Node-Admittance Matrix $\mathbf{Y}_n$ By Inspection

Node – Votage Equation :

$$\underbrace{\begin{bmatrix} Y_{11} & Y_{12} & \cdots & Y_{1,n-1} \\ Y_{21} & Y_{22} & \cdots & Y_{2,n-1} \\ \vdots & \vdots & \cdots & \vdots \\ Y_{n-1,1} & Y_{n-1,2} & \cdots & Y_{n-1,n-1} \end{bmatrix}}_{\mathbf{Y}_n} \underbrace{\begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_{n-1} \end{bmatrix}}_{\mathbf{e}} = \underbrace{\begin{bmatrix} i_{s_1} \\ i_{s_2} \\ \vdots \\ i_{s_{n-1}} \end{bmatrix}}_{\mathbf{i}_s} \qquad (8)$$

where

i_{s_j} = - (algebraic sum of all current sources leaving node (j))

Diagonal Elements of $\mathbf{Y}_n$

Y_{mm} = sum of admittances $Y_j \triangleq \dfrac{1}{R_j}$ of all resistors connected to node (m) , $m = 1, 2, \ldots, n-1$, where n is the total number of nodes.

$\mathbf{Y}_n$ is called the node-admittance matrix.

$\mathbf{e}$ is the node-to-datum voltage vector.

$\mathbf{i}_s$ is called the node current source vector.

Off-Diagonal Elements of $\mathbf{Y}_n$

Y_{jk} = - (sum of admittances $Y_j \triangleq \dfrac{1}{R_j}$ of all resistors connected across node (j) and node (k))

Symmetry Property:

$\mathbf{Y}_n$ is a symmetric matrix, i.e., $\boxed{Y_{jk} = Y_{kj}}$

Proof :

Since $\mathbf{Y}_b$ in (1) is a diagonal matrix,

$$\mathbf{Y}_b = \mathbf{Y}_b{}^T$$

$$\begin{aligned}\mathbf{Y}_n^T &= \left(\mathbf{A}\mathbf{Y}_b\mathbf{A}^T\right)^T \\ &= \mathbf{A}\mathbf{Y}_b^T\mathbf{A}^T \\ &= \mathbf{A}\mathbf{Y}_b\mathbf{A}^T \\ &= \mathbf{Y}_n \qquad \blacksquare\end{aligned}$$

Writing Mesh-Impedance Matrix Z_m By Inspection

Let m be the total number of meshes of a planar digraph G, including the exterior mesh formed by traversing the outer boundary branches (i.e., those branches having only one circulating current $\hat{i}_j$ passing through them). Hence,

$m =$ number of interior meshes (windows) +1

Mesh-Current Equation :

$$\underbrace{\begin{bmatrix} Z_{11} & Z_{12} & \cdots & Z_{1,m-1} \\ Z_{21} & Z_{22} & \cdots & Z_{2,m-1} \\ \vdots & \vdots & \cdots & \vdots \\ Z_{m-1,1} & Z_{m-1,2} & \cdots & Z_{m-1,m-1} \end{bmatrix}}_{\mathbf{Z}_m} \underbrace{\begin{bmatrix} \hat{i}_1 \\ \hat{i}_2 \\ \vdots \\ \hat{i}_{m-1} \end{bmatrix}}_{\hat{\mathbf{i}}} = \underbrace{\begin{bmatrix} v_{s_1} \\ v_{s_2} \\ \vdots \\ v_{s_{m-1}} \end{bmatrix}}_{\mathbf{v}_s} \quad (1)$$

where

$v_{s_j} = -$ (clockwise algebraic sum of all voltage sources around mesh j)

Diagonal Elements of $\mathbf{Z}_m$

Z_{kk} = sum of impedances $Z_j \triangleq R_j$ of all resistors located along mesh "k", $k=1, 2, \cdots, m-1$, where m is the total number of (interior and exterior) meshes. (2)

$\mathbf{Z}_m$ is called the mesh-impedance matrix.

$\hat{\mathbf{i}}$ is called the mesh-current vector.

$\mathbf{v}_s$ is called the mesh-voltage source vector.

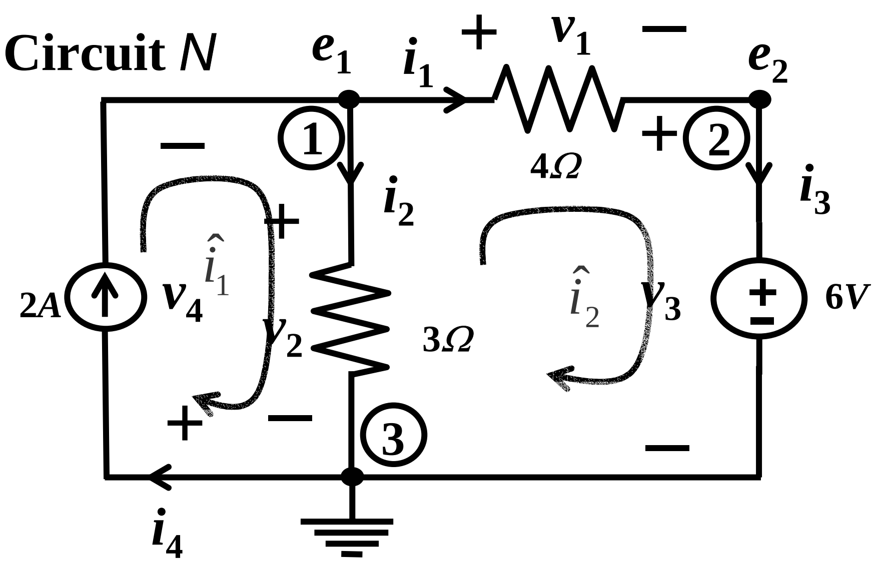
Off-Diagonal Elements of $\mathbf{Z}_m$

$Z_{jk} = -$(sum of impedances $Z_j \triangleq R_j$ of all resistors along both mesh j and mesh k) (3)

Symmetry Property:

$\mathbf{Z}_m$ is a symmetric matrix, i.e., $Z_{jk} = Z_{kj}$ (4)

Extended Mesh Current Method



$$v_1 = 4\hat{i}_2$$

$$v_2 = 3\left(\hat{i}_1 - \hat{i}_2\right)$$

Step 1.

When the circuit contains "β" current sources $i_{s_1}, i_{s_2}, \cdots, i_{s_\beta}$, use their associated voltages $v_{s_1}, v_{s_2}, \cdots, v_{s_\beta}$ when applying KVL.

KVL around mesh 1: $$3\left(\hat{i}_1 - \hat{i}_2\right) + v_4 = 0 \qquad (1)$$

KVL around mesh 2: $$-3\left(\hat{i}_1 - \hat{i}_2\right) + 4\hat{i}_2 = -6 \qquad (2)$$

Step 2.

For each current source i_{s_j}, add an equation $i_{s_j}^+ - i_{s_j}^- = i_{s_j}$.

$$\hat{i}_1 = 2 \qquad (3)$$

Step 3.

Solve the $(m\text{-}1) + \beta$ equations for $\hat{i}_1, \hat{i}_2, \cdots \hat{i}_{m-1}, v_{s_1}, v_{s_2}, \cdots v_{s_\beta}$.

Substituting (3) into (2), we obtain :

$$-3\left(2 - \hat{i}_2\right) + 4\hat{i}_2 = -6 \Rightarrow \boxed{\hat{i}_2 = 0} \qquad (4)$$

Substituting (3) and (4) into (1), we obtain :

$$3(2 - 0) + v_4 = 0 \Rightarrow \boxed{v_4 = -6V} \qquad (5)$$

Note:

The unknown variables in the extended mesh current method consist of the usual *m*-1 mesh currents, plus the unknown voltages associated with the current sources.

Hence, if there are "β" current sources, the extended mesh current method would consist of (*m*-1)+β independent linear equations involving (*m*-1)+β unknown variables

$$\Big\{ \underbrace{\hat{i}_1, \hat{i}_2, \cdots, \hat{i}_{m-1}}_{(m\text{-}1)\text{ mesh current variables}}, \underbrace{v_{s_1}, v_{s_2}, \cdots v_{s_\beta}}_{\beta\text{ voltage variables}} \Big\}.$$

All branch voltages and currents can be trivially calculated from $\hat{i}_2$ and v_4.

$$i_1 = \hat{i}_2 = 0\ A\,, \qquad v_1 = 4\,i_1 = 0\,V$$

$$i_2 = \hat{i}_2 - \hat{i}_1 = 2\ A\,, \qquad v_2 = 3\,i_2 = 6\,V$$

$$i_3 = \hat{i}_2 = 0\ A\,, \qquad v_3 = 6\,V$$

$$i_4 = \hat{i}_1 = 2\ A\,, \qquad v_4 = -6\,V$$

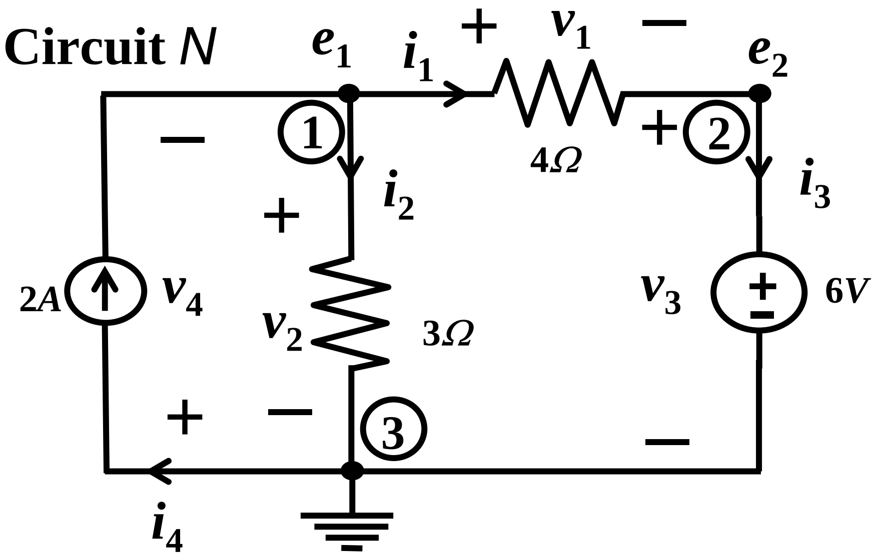
Verification of Solution by Tellegen's Theorem :

$$\sum_{j=1}^{4} v_j\, i_j = (v_1\, i_1) + (v_2\, i_2) + (v_3\, i_3) + (v_4\, i_4)$$

$$= (0)(0) + (6)(2) + (6)(0) + (-6)(2)$$

$$\stackrel{?}{=} 0$$

Extended Node Voltage Method



$$i_1 = \frac{e_1 - e_2}{4}$$

$$i_2 = \frac{e_1}{3}$$

Step 1.

When the circuit contains "α" voltages $v_{s_1}, v_{s_2}, \cdots, v_{s_\alpha}$, use their associated currents $i_{s_1}, i_{s_2}, \cdots, i_{s_\alpha}$ when applying KCL.

KCL at ①: $$\frac{e_1}{3} + \frac{(e_1 - e_2)}{4} = 2 \qquad (1)$$

KCL at ②: $$-\frac{(e_1 - e_2)}{4} + i_3 = 0 \qquad (2)$$

Step 2.

For each voltage source v_{s_j}, add an equation $e_j^+ - e_j^- = v_{s_j}$.

$$e_2 = 6 \qquad (3)$$

Step 3.

Solve the $(n\text{-}1) + \alpha$ equations for $e_1, e_2, \cdots e_{n-1}, i_{s_1}, i_{s_2}, \cdots i_{s_\alpha}$.

Substituting (3) into (1), we obtain :

$$\frac{e_1}{3} + \frac{(e_1 - 6)}{4} = 2 \quad \Rightarrow \quad \boxed{e_1 = 6V} \qquad (4)$$

Substituting (4) into (2), we obtain :

$$\boxed{i_3 = 0} \qquad (5)$$

Note:

The unknown variables in the extended node voltage method consist of the usual n-1 node-to-datum voltages, plus the unknown currents associated with the voltage sources.

Hence, if there are "α" voltage sources, the modified node voltage method would consist of (n-1)+α independent linear equations involving (n-1)+α unknown variables

$$\{\underbrace{e_1, e_2, \cdots, e_{n-1}}_{(n\text{-}1)\text{ node-to-datum voltage variables}}, \underbrace{i_{s_1}, i_{s_2}, \cdots i_{s_\alpha}}_{\alpha\text{ current variables}}\}.$$

Sufficient Condition for G to be Planar

If G has less than 9 branches, it is planar.

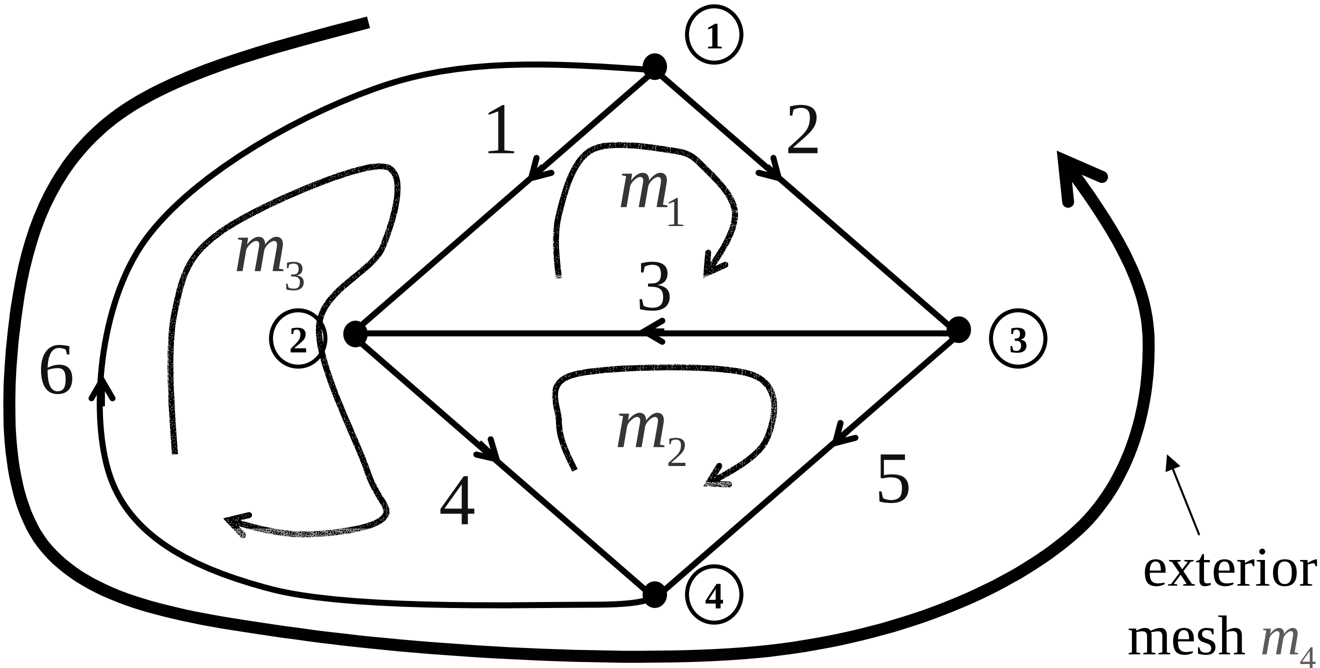
***Proof*.**

The 2 basic nonplanar graphs have 9 and 10 branches, respectively.

Mapping a planar digraph on a sphere

A digraph *G* is **planar** if, and only if, it can be drawn on the surface of a sphere such that *G* can be partitioned into contiguous regions and colored (as in a map of countries) such that no two regions have overlapping colors.

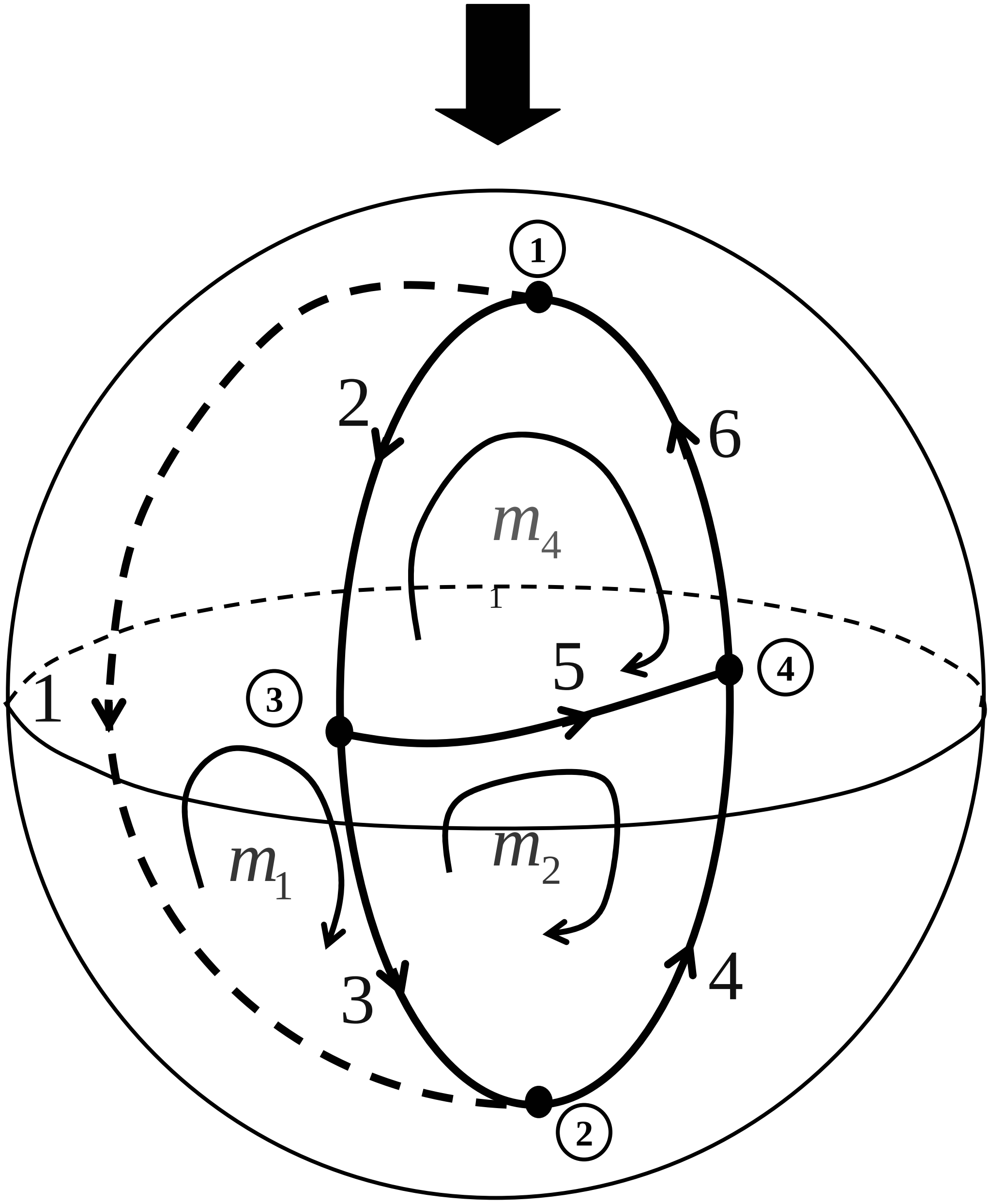
Mapping a planar digraph on a sphere



m_1, m_2, and m_3 are **interior clockwise meshes**.

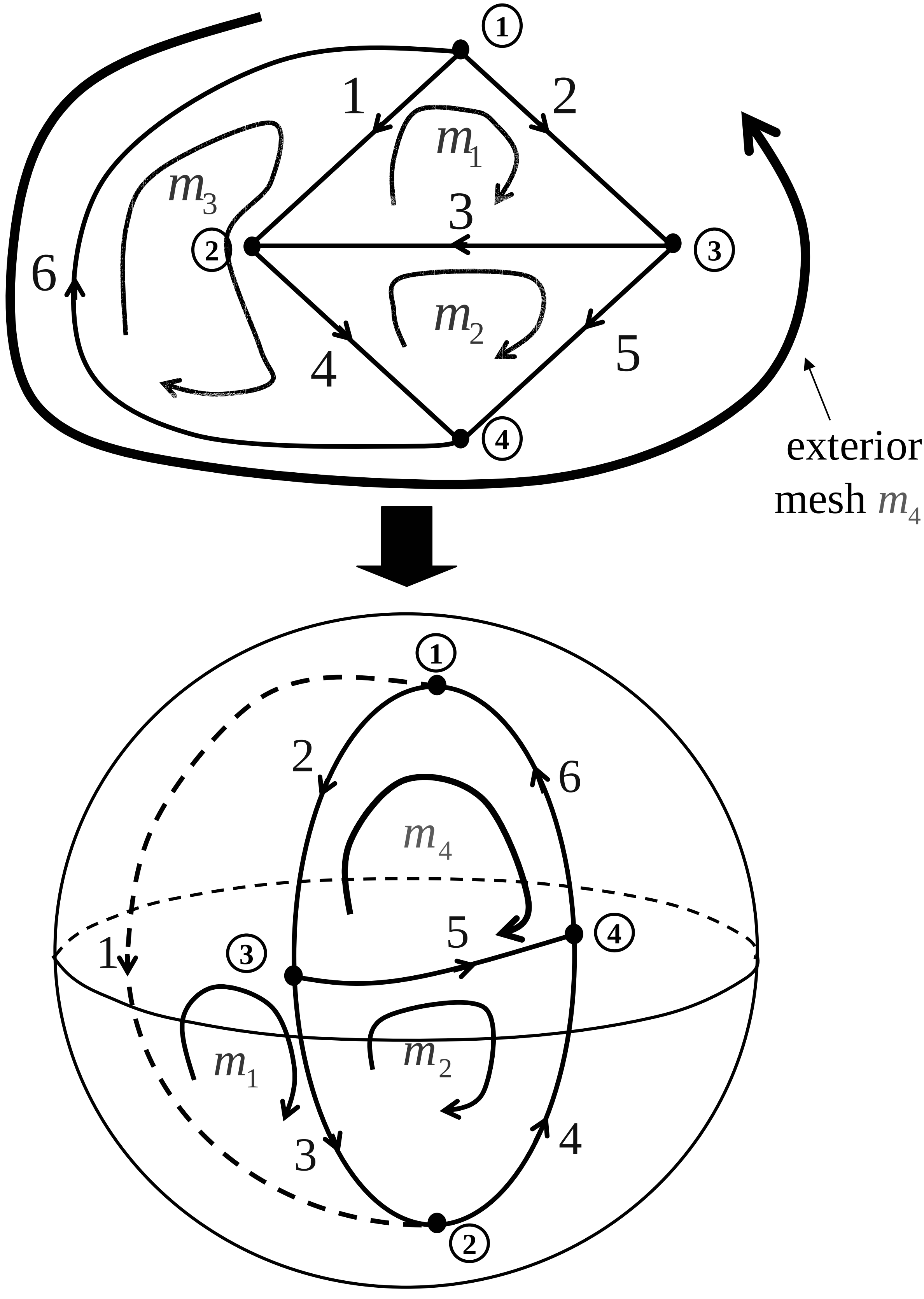
We can always redraw a ***planar*** digraph on the surface of a **sphere** without interior branches and vice-versa, as illustrated below.

- Although mesh m_3 (formed by tracing along branches {1, 4, 6} in the above planar digraph in a clockwise direction) appears to be **counterclockwise** on the sphere, it actually moves in a clockwise direction when viewed from behind.
- Although the counterclockwise loop m_4 formed by **exterior** branches (i.e, boundary branches with only 1 circulating current) {6, 5, 2} does not look like a mesh on the planar digraph, it is in fact a **clockwise** mesh when the digraph is mapped on the surface of a sphere.



Just as all countries on a **globe** can be mapped onto a flat plane of paper, any **planar** digraph drawn on a **sphere** can always be redrawn as a **planar** digraph on a plane.

Mapping a planar digraph on a sphere

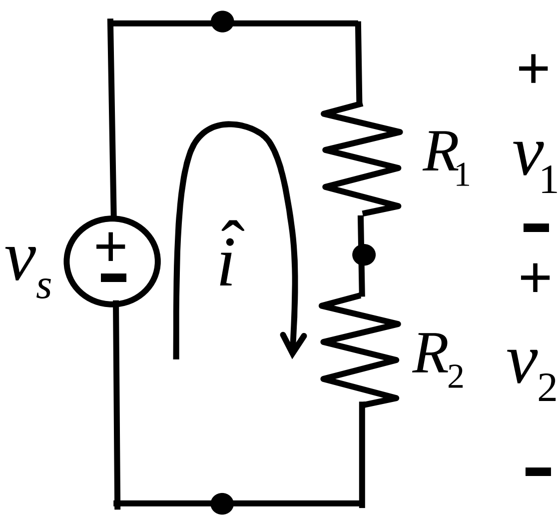


Duality Principle

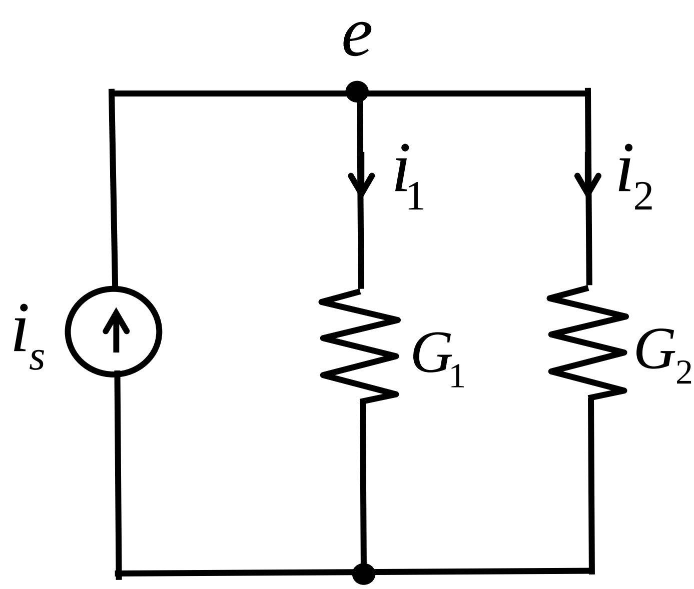
There are many physical variables, concepts, properties, and theorems in electrical circuit theory which appear in pairs, henceforth called **dual pairs**, such that for each circuit theorem, property, concept, etc., there is a **dual theorem**, **dual property**, and **dual concept**, respectively. This duality principle is extremely useful since we only need to learn and memorize half of them!

Variable, concept, property	Dual variable, concept, property
KVL	KCL
KCL	KVL
Voltage	Current
Current	Voltage
Series	Parallel
Parallel	Series
Resistance (Ohms)	Conductance (Siemens)
Impedance	Admittance
Admittance	Impedance
Node (non-datum)	Mesh (interior)
Node voltage	Mesh current
Datum node	Exterior mesh

Example of Duality



series circuit



parallel circuit

KVL : $(R_1 + R_2)\hat{i} = v_s$

voltage divider:

$$v_2 = \left(\frac{R_2}{R_1 + R_2} \right) v_s$$

KCL : $(G_1 + G_2)e = i_s$

current divider:

$$i_2 = \left(\frac{G_2}{G_1 + G_2} \right) i_s$$

Duality Theorem

For planar circuits, the **node-voltage method** and the **mesh-current method**, as well as their extended versions, are dual sets of equations which can be derived from each other via their **dual** variables.

Node-voltage Equation	Mesh-current equation
$\mathbf{Y}_n \mathbf{e} = \mathbf{i}_s$	$\mathbf{Z}_m \hat{\mathbf{i}} = \mathbf{v}_s$
Node-Admittance matrix	**Mesh-Impedance matrix**
$\mathbf{Y}_n$	$\mathbf{Z}_m$