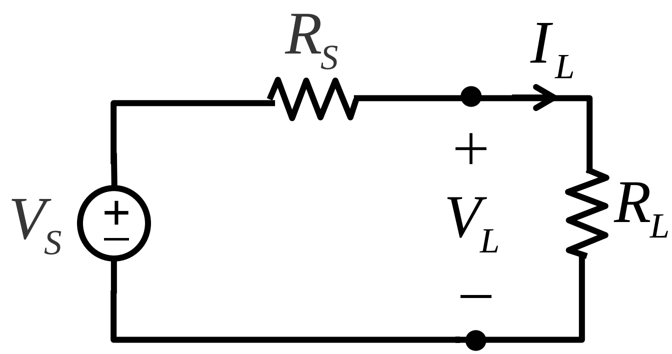
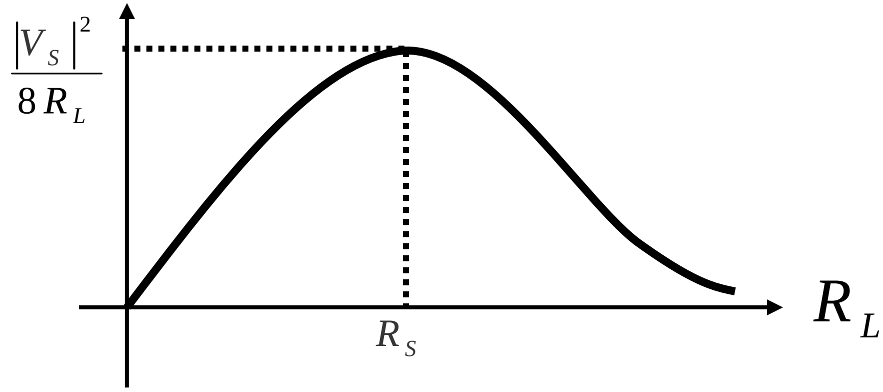
Maximum Power Transfer



$$I_L = \frac{V_S}{R_S + R_L}$$

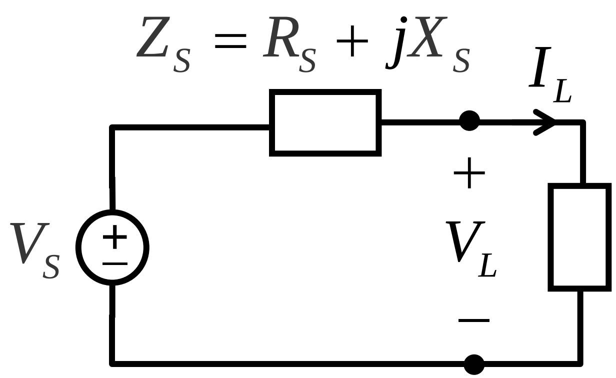
$$Pav_L = \frac{1}{2}|V_L||I_L| = \frac{1}{2}R_L|I_L|^2 = \frac{1}{2}|V_S|^2\frac{R_L}{(R_S+R_L)^2}$$



∴ For fixed V_s and R_s, maximum average power transfer to load R_L occurs when

$$\frac{dPav_L}{\partial R_L} = \frac{1}{2}|V_S|^2\frac{(R_S+R_L)^2 - 2(R_S+R_L)R_L}{(R_S+R_L)^4} = 0 \Rightarrow \boxed{R_L = R_S}$$

Maximum Power Transfer



$$I_L = \frac{V_S}{Z_S + Z_L}$$

$$Z_L = R_L + jX_L \;,\; R_L > 0$$

$$V_L = Z_L I_L$$

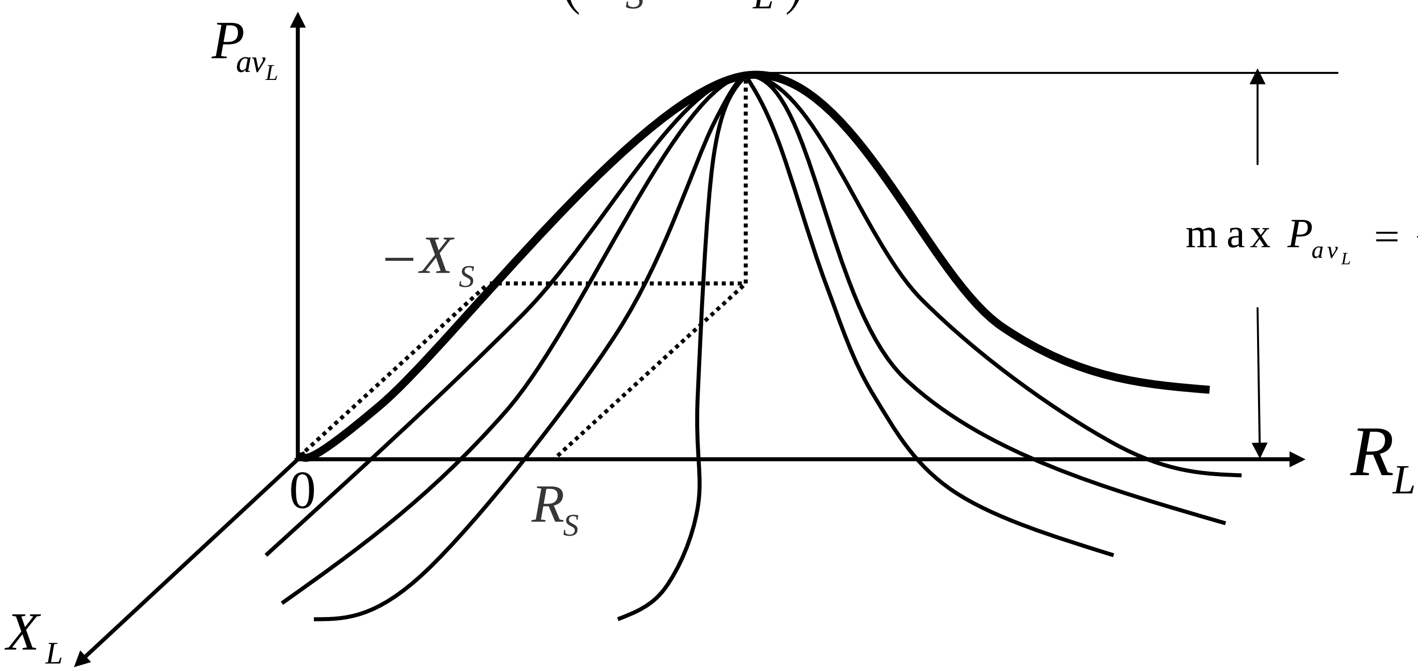
$$Pav_L = \frac{1}{2}|V_L||I_L|\cos\measuredangle Z_L = \frac{1}{2}|Z_L||I_L|^2\cos\measuredangle Z_L = \frac{1}{2}R_L|I_L|^2$$

$$= \frac{1}{2}|V_S|^2\frac{R_L}{|Z_S + Z_L|^2} = \frac{1}{2}|V_S|^2\frac{R_L}{(R_S + R_L)^2 + (X_S + X_L)^2}$$

To maximize P_{av_L}, necessary to set $\boxed{X_L = -X_S}$

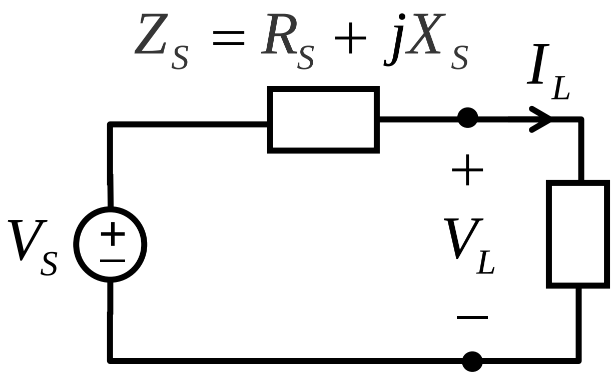
$$\Rightarrow \quad P_{av_L} = \frac{1}{2}\left|V_S\right|^2 \frac{R_L}{\left(R_S + R_L\right)^2}$$

$$\frac{dPav_L}{\partial R_L} = \frac{1}{2}\left|V_S\right|^2 \frac{\left(R_S + R_L\right)^2 - 2\left(R_S + R_L\right)R_L}{\left(R_S + R_L\right)^4} = 0 \Rightarrow \boxed{R_L = R_S}$$



Theorem : Maximum Power Theorem

Optimum Load Impedance $Z_{L_{opt}} = \overline{Z}_S$



$$I_L = \frac{V_S}{Z_S + Z_L}$$

$$Z_L = R_L + jX_L \, , \, R_L > 0$$

Max. Power Theorem : $Z_{L_{opt}} = \bar{Z}_S$ **Conjugate-match condition**

$$Pav_S = \frac{1}{2}\left|V_S\right|\left|I_S\right|\cos\measuredangle\left(Z_S + Z_L\right)$$

$$= \frac{1}{2}\left(\left|\left(Z_S + Z_L\right)\right| \bullet \left|I_S\right|\right)\left|I_S\right|\cos\measuredangle\left(Z_S + Z_L\right)$$

$$= \frac{1}{2}\left|I_S\right|^2 \mathrm{Re}\left(Z_S + Z_L\right)$$

$$= \frac{1}{2}\left|I_L\right|^2 \mathrm{Re}\left(Z_S + Z_L\right)$$

$$= \frac{1}{2}\frac{\left|V_S\right|^2}{\left|Z_S + Z_L\right|^2} \mathrm{Re}\left(Z_S + Z_L\right)$$

$$\therefore \quad Pav_S \Big|_{Z_L = \overline{Z}_S} = \frac{1}{2}\frac{|V_S|^2}{|Z_S + \overline{Z}_S|^2}\mathrm{Re}\left(Z_S + \overline{Z}_S\right) = \frac{1}{2}\frac{|V_S|^2}{|2R_S|^2}\left(2R_S\right)$$

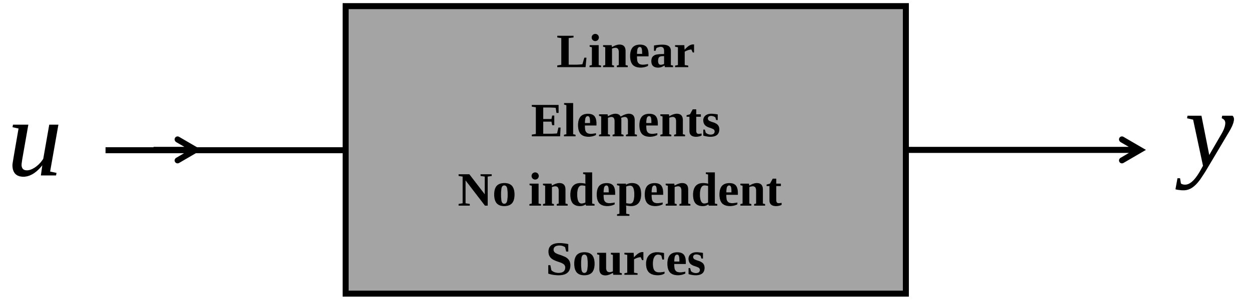
$$= \frac{|V_S|^2}{4R_S} = 2P_{av_L}$$

$$\text{Efficiency} = \frac{P_{av_L}}{P_{av_S}} = \frac{P_{av_L}}{2P_{av_L}} = \frac{1}{2} \quad \text{or} \quad 50\%$$

Comments :

1. Under conjugate-match condition, 50% of the power delivered by the source is lost as heat dissipation in R_S. $\therefore$ Power company **never** conjugates their loads!
2. **Max. Power Theorem** is used extensively in communication circuits to extract maximum power from preceding stages.

Network Functions



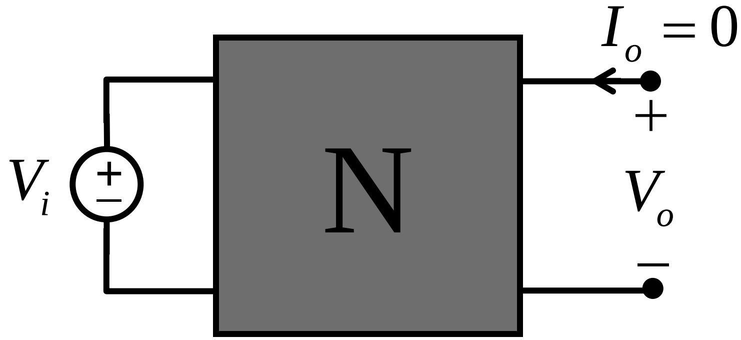
$$u(t) = |U| \cos(\omega t + \measuredangle U) \quad \Updownarrow \quad U = |U| e^{j \measuredangle U}$$

$$y(t) = |Y| \cos(\omega t + \measuredangle Y) \quad \Updownarrow \quad Y = |Y| e^{j \measuredangle Y}$$

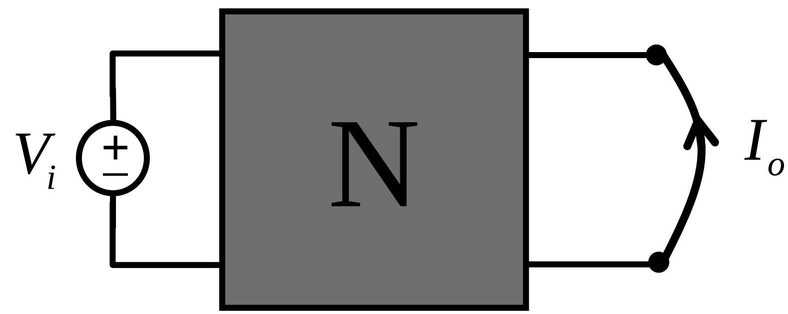
Definition

$$H(j\omega) \triangleq \frac{Y(j\omega)}{U(j\omega)}$$

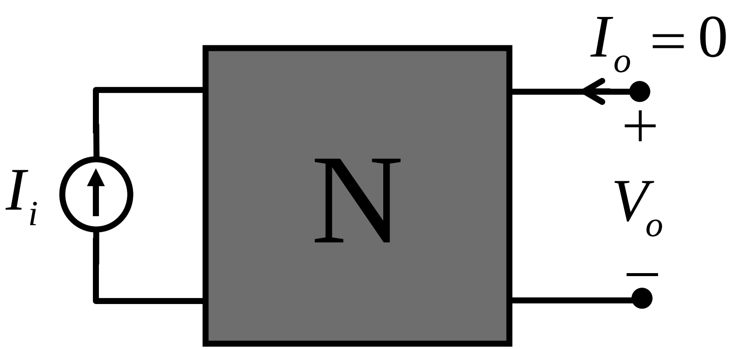
is called a **Network Function**.



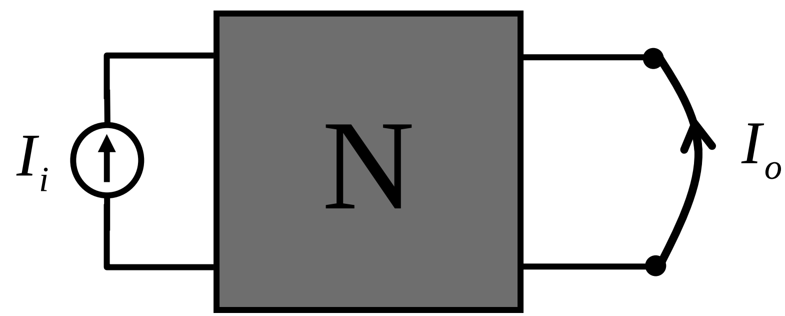
$$H(j\omega) \triangleq \frac{V_o(j\omega)}{V_i(j\omega)}$$



$$H(j\omega) \triangleq \frac{I_o(j\omega)}{V_i(j\omega)}$$

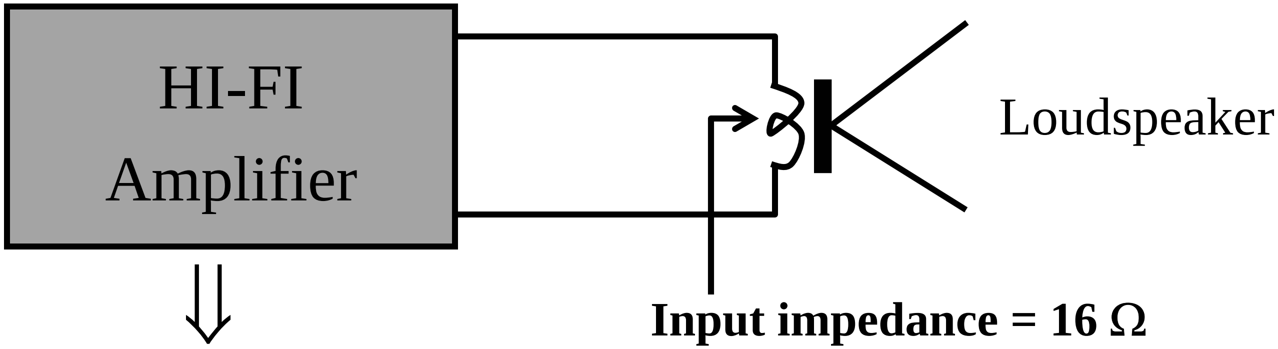


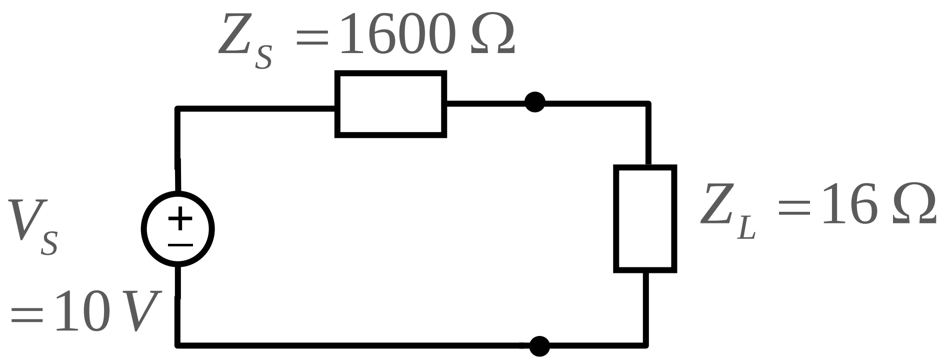
$$H(j\omega) \triangleq \frac{V_o(j\omega)}{I_i(j\omega)}$$



$$H(j\omega) \triangleq \frac{I_o(j\omega)}{I_i(j\omega)}$$

Typical Application of Max. Power Theorem





Let P_o = average power delivered to loudspeaker

$$P_o = \frac{1}{2}\left|V_S\right|^2 \frac{\operatorname{Re} Z_L}{\left|Z_S + Z_L\right|^2} = \frac{1}{2}(10)^2\left(\frac{16}{1600+16}\right) \approx 50\left(10^{-2}\right) = 0.5\,W$$

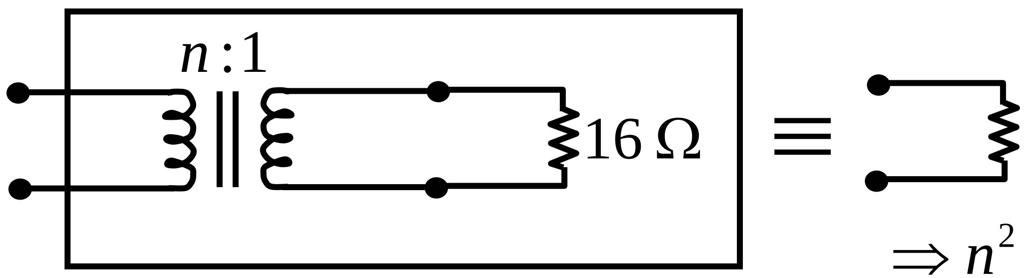
For maximum power transfer, make $Z_L = 1600\,\Omega$.

$$P_o = \frac{1}{2}(10)^2\left(\frac{1600}{1600+1600}\right) = 25\,W$$

For maximum power transfer, make $Z_L = 1600\,\Omega$.

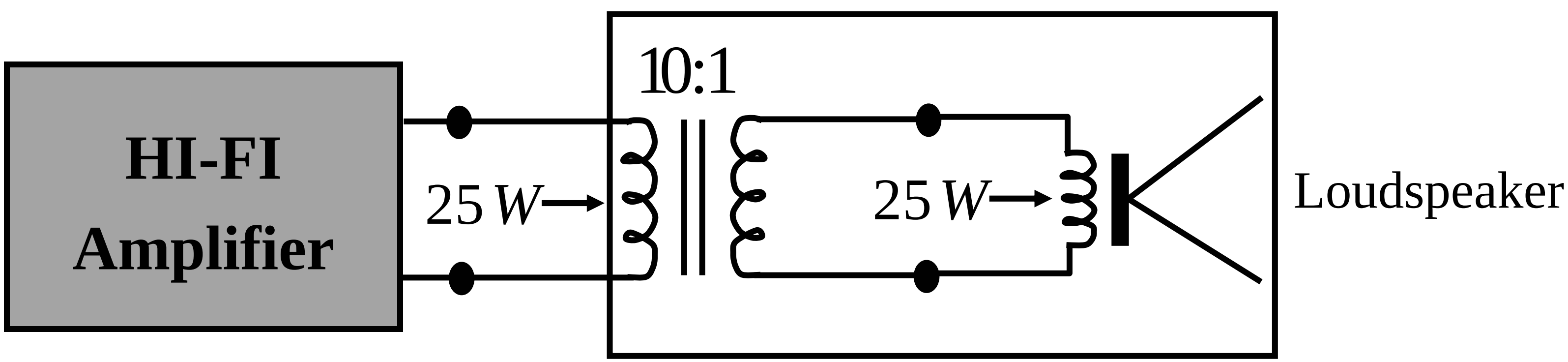
$$P_o = \frac{1}{2}(10)^2\left(\frac{1600}{1600+1600}\right) = 25\,W$$

Use a transformer :



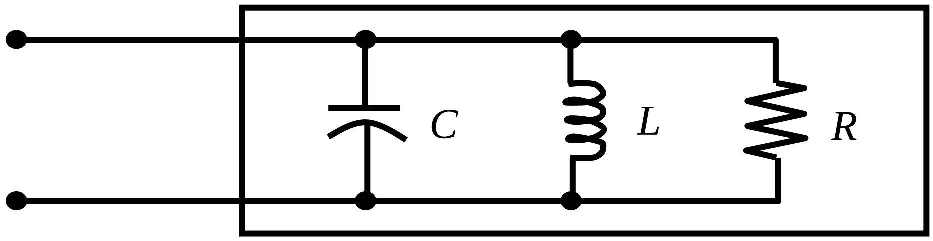
$$Z_L = n^2(16) = 1600$$

$$\Rightarrow n^2 = 100 \quad \Rightarrow n = 10$$



transformer is non-energic $\Rightarrow$ 25 Watts of power is delivered to loudspeaker.

FREQUENCY RESPONSE



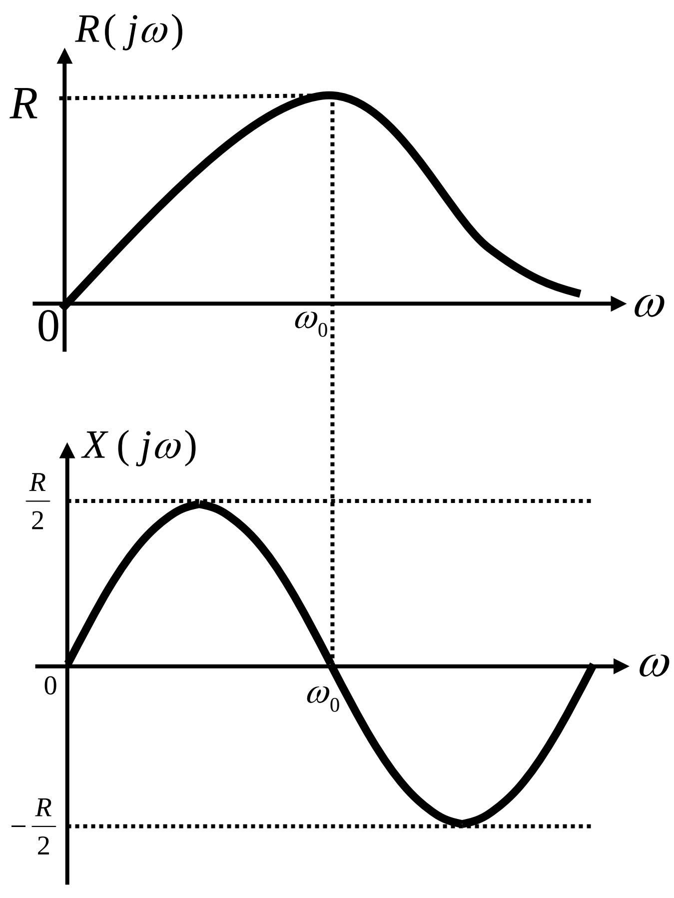
$$Z(j\omega) = \underbrace{\frac{1/R}{\left(1/R\right)^2 + \left(\omega C - 1/\omega L\right)^2}}_{R(j\omega)} + j\,\underbrace{\frac{-\left(\omega C - 1/\omega L\right)}{\left(1/R\right)^2 + \left(\omega C - 1/\omega L\right)^2}}_{X(j\omega)}$$

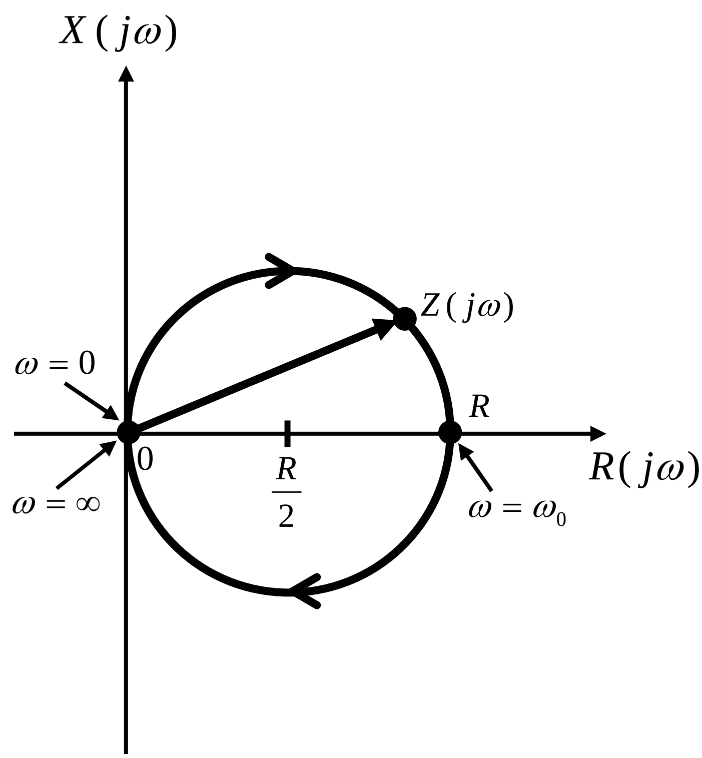
Resistance function $\rightarrow R(j\omega)$

$X(j\omega)$

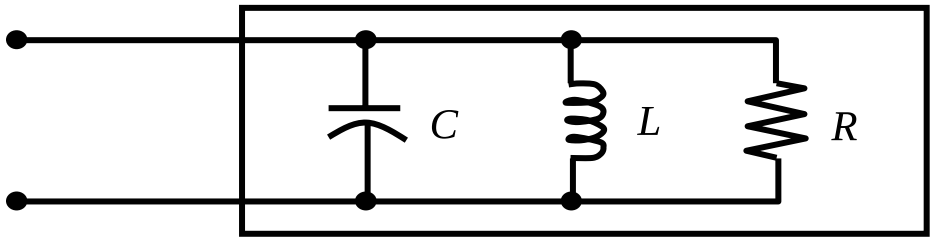
$\leftarrow$ Reactance function

$$\omega C = 1/\omega L \Rightarrow \omega = 1/\sqrt{LC} \triangleq \omega_0 \leftarrow$$ **Resonant frequency**





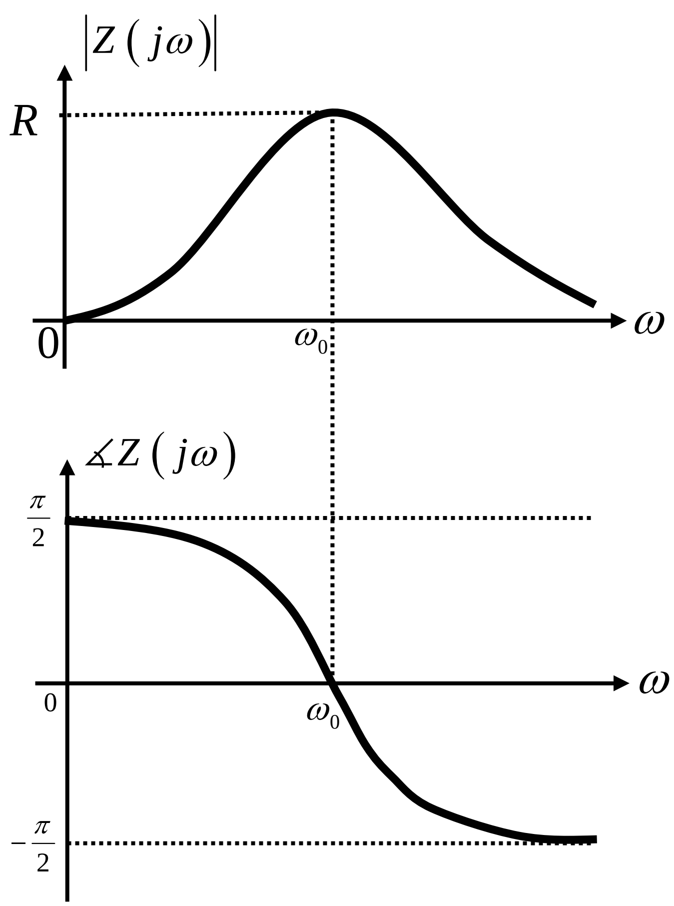
FREQUENCY RESPONSE

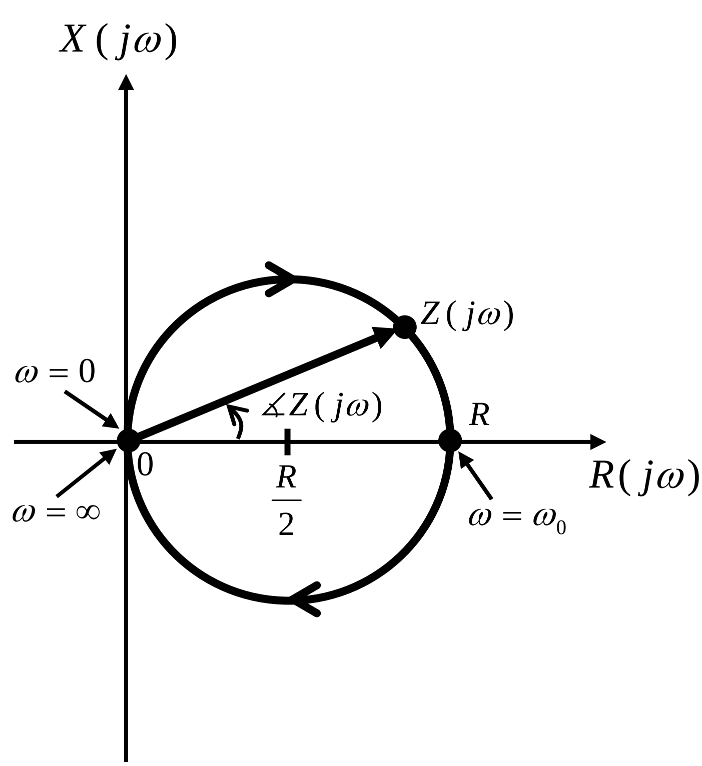


$$\left|Z\left(j\omega\right)\right| = \sqrt{\left(\frac{1/R}{\left(1/R\right)^2 + \left(\omega C - 1/\omega L\right)^2}\right)^2 + \left(\frac{\omega C - 1/\omega L}{\left(1/R\right)^2 + \left(\omega C - 1/\omega L\right)^2}\right)^2}$$

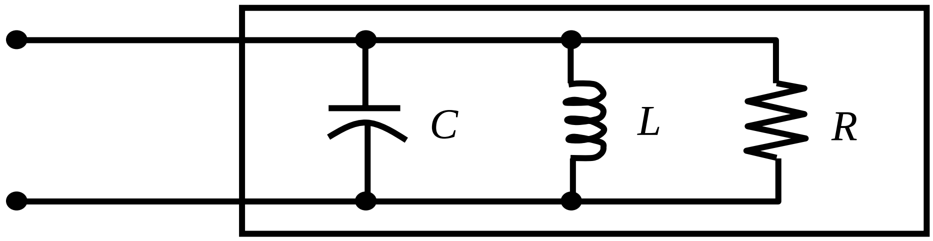
$$\measuredangle Z\left(j\omega\right) = \tan^{-1}\left(\frac{-\left(\omega C - 1/\omega L\right)}{1/R}\right)$$

$$\omega C = 1/\omega L \Rightarrow \omega = 1/\sqrt{LC} \triangleq \omega_0 \leftarrow$$ **Resonant frequency**





FREQUENCY RESPONSE

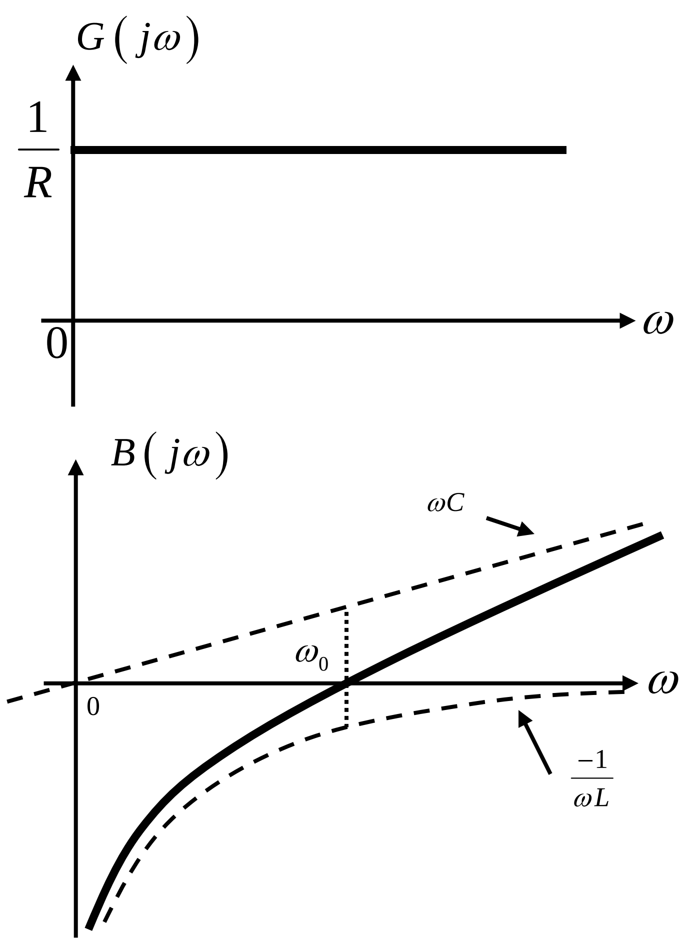


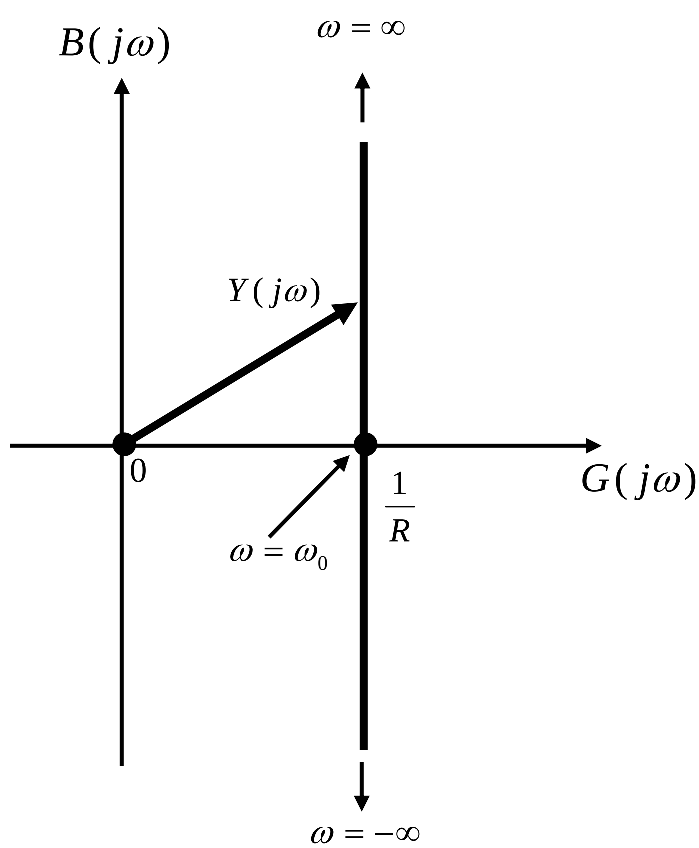
$$Y(j\omega) = \underbrace{1/R}_{G(j\omega)} + j\underbrace{\left(\omega C - 1/\omega L\right)}_{B(j\omega)}$$

$G(j\omega)$ ↑ Conductance function

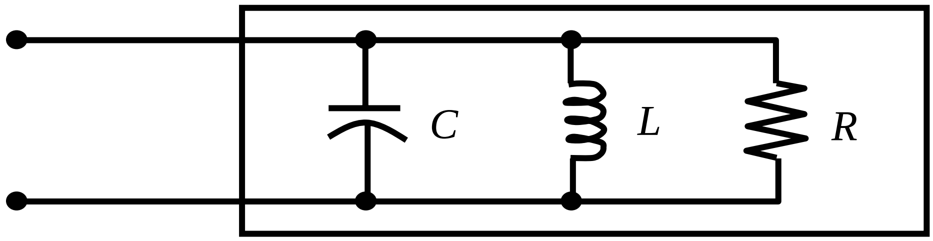
$B(j\omega)$ ← Susceptance function

$$\omega C = 1/\omega L \Rightarrow \omega = 1/\sqrt{LC} \triangleq \omega_0$$ ← **Resonant frequency**





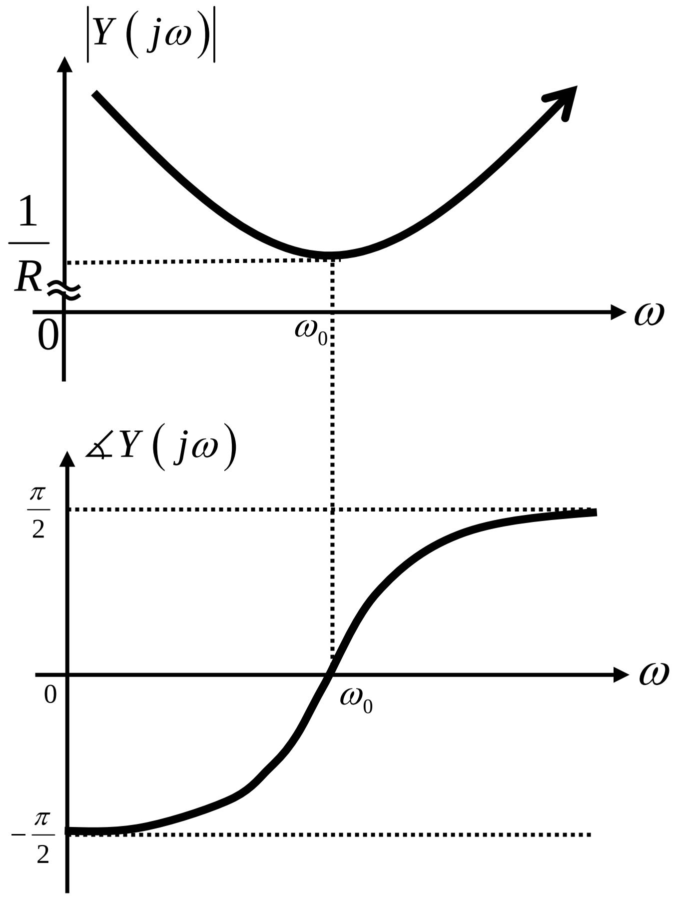
FREQUENCY RESPONSE

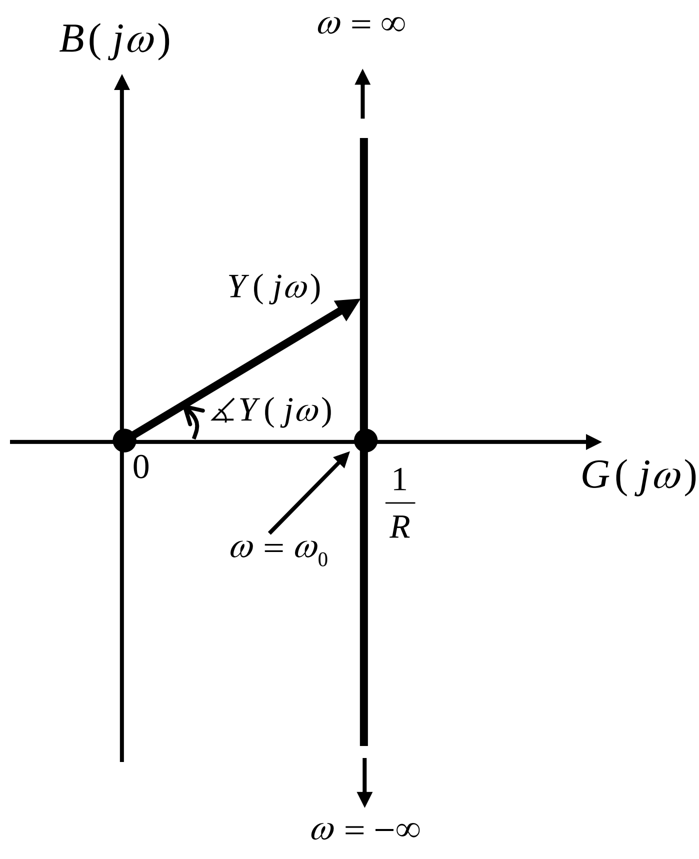


$$\left|Y(j\omega)\right| = \sqrt{\left(\tfrac{1}{R}\right)^2 + \left(\omega C - \tfrac{1}{\omega L}\right)^2} \quad \leftarrow \text{ Magnitude function}$$

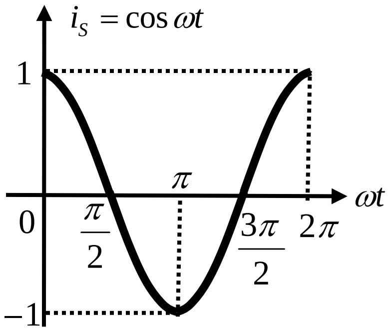
$$\measuredangle Y(j\omega) = \tan^{-1}\left(\frac{\omega C - \tfrac{1}{\omega L}}{\tfrac{1}{R}}\right) \quad \leftarrow \text{ Phase function}$$

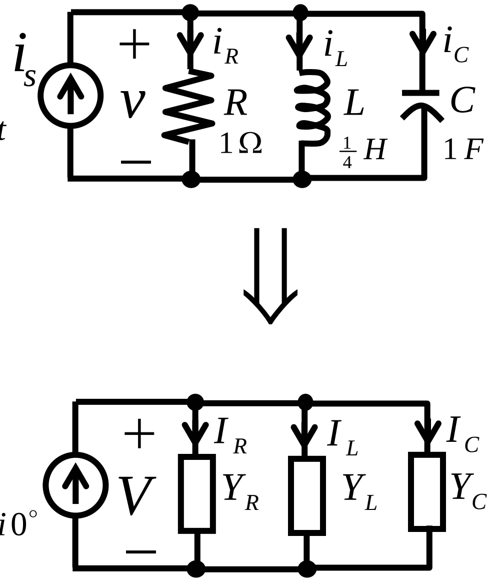
$$\omega C = \tfrac{1}{\omega L} \Rightarrow \omega = \tfrac{1}{\sqrt{LC}} \triangleq \omega_0 \leftarrow \boxed{\text{Resonant frequency}}$$





Resonance

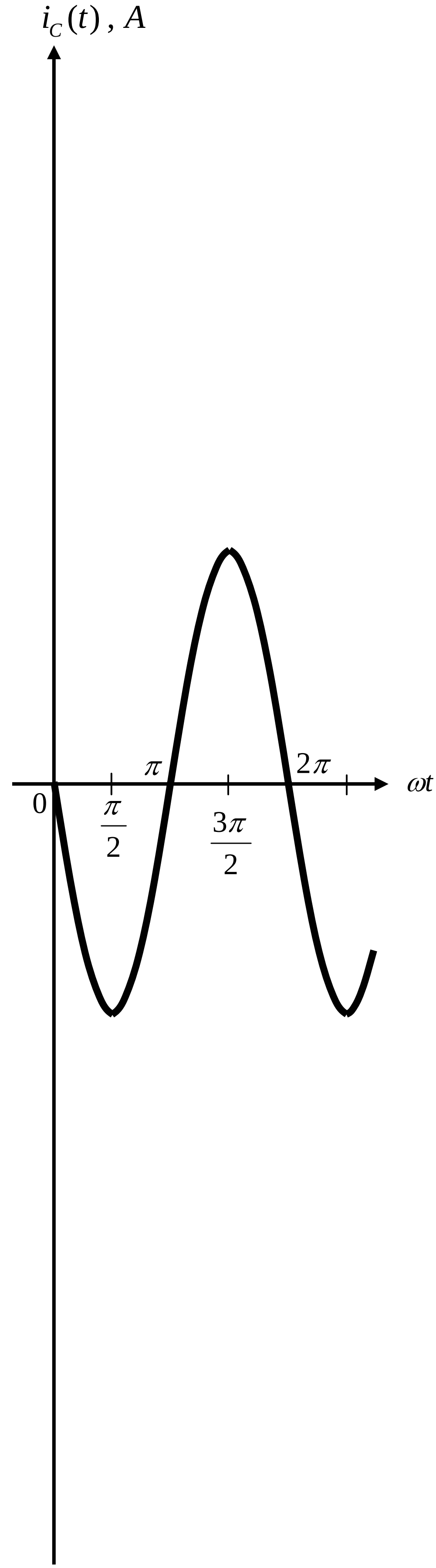




$$\omega = 2$$

$$\omega_0 = \frac{1}{\sqrt{LC}} = 2$$

$$I = 1e^{i0^\circ}$$



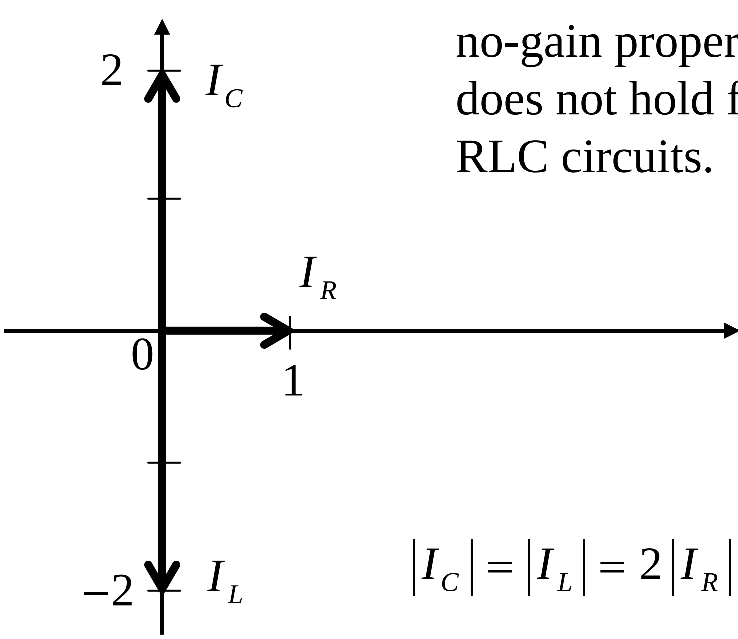
$$Y(j2) = 1 \qquad , Z(j2) = 1$$

$$V(j2) = Z(j2)I(j2) = 1$$

$$I_R(j2) = Y_R(j2)V(j2) = 1$$

$$I_L(j2) = Y_L(j2)V(j2) = 2\angle -90^\circ$$

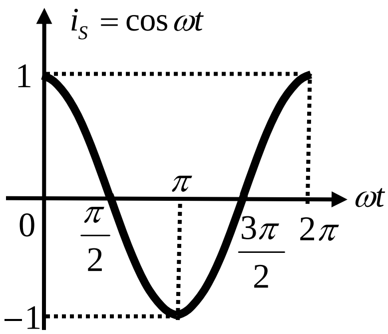
$$I_C(j2) = Y_C(j2)V(j2) = 2\angle 90^\circ$$

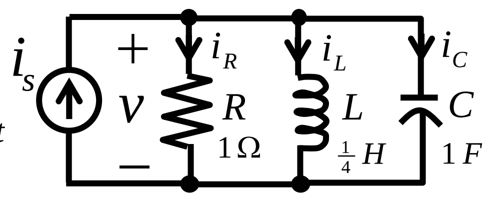


no-gain property does not hold for RLC circuits.

$$|I_C| = |I_L| = 2|I_R|$$

Resonance

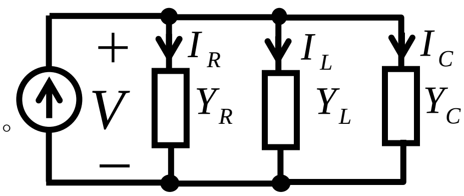




$\Downarrow$

$$\omega = 1$$

$$\omega_0 = \frac{1}{\sqrt{LC}} = 2$$



$$I = 1e^{i0^\circ}$$

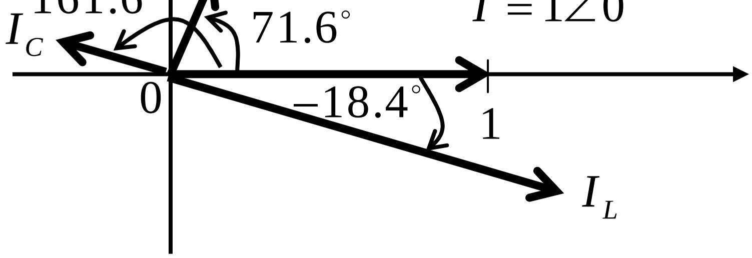
$$Y(j1) = \sqrt{10}\angle -71.6^\circ \;,\; Z(j1) = \frac{1}{\sqrt{10}}\angle 71.6^\circ$$

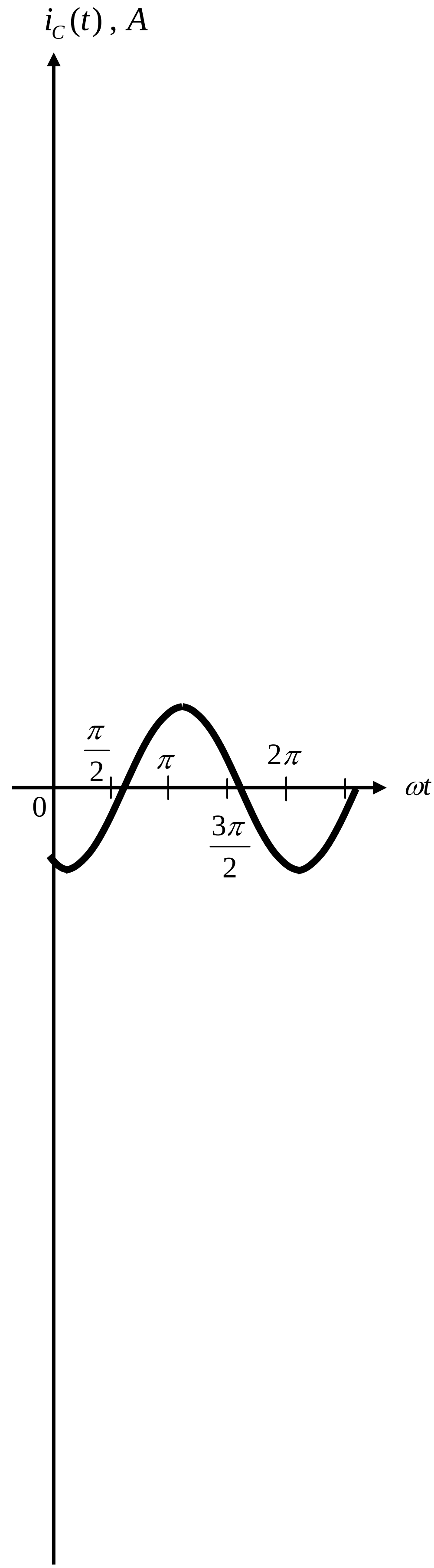
$$V(j1) = Z(j1)I(j1) = \frac{1}{\sqrt{10}}\angle 71.6^\circ$$

$$I_R(j1) = Y_R(j1)V(j1) = \frac{1}{\sqrt{10}}\angle 71.6^\circ$$

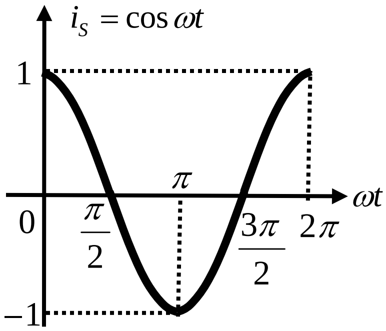
$$I_L(j1) = Y_L(j1)V(j1) = \frac{4}{\sqrt{10}}\angle -18.4^\circ$$

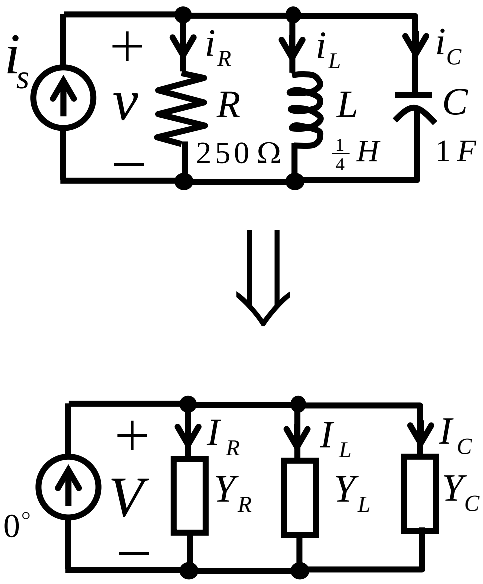
$$I_C(j1) = Y_C(j1)V(j1) = \frac{1}{\sqrt{10}}\angle 161.6^\circ$$





Resonance





$$\omega = 2$$

$$\omega_0 = \frac{1}{\sqrt{LC}} = 2$$

$$I = 1e^{i0^\circ}$$

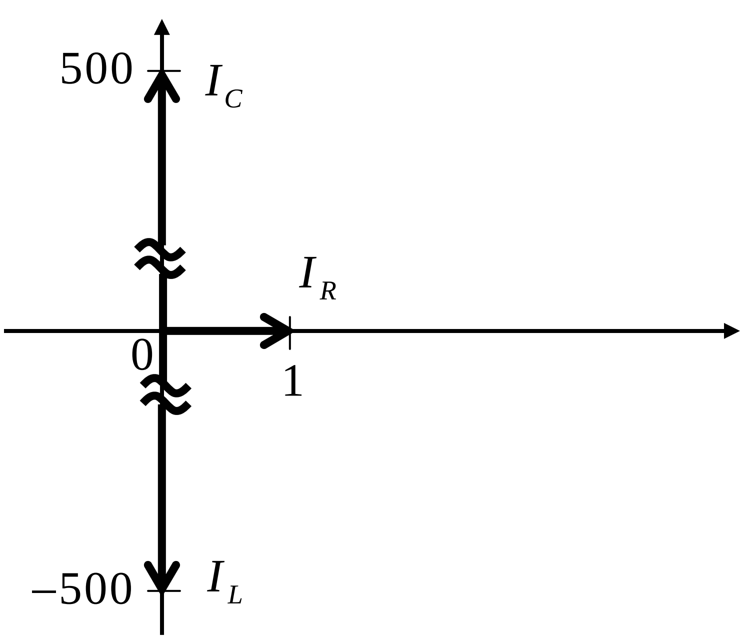
$$Y(j2) = 4\times 10^{-3}\,℧ \quad , Z(j2) = 250\,\Omega$$

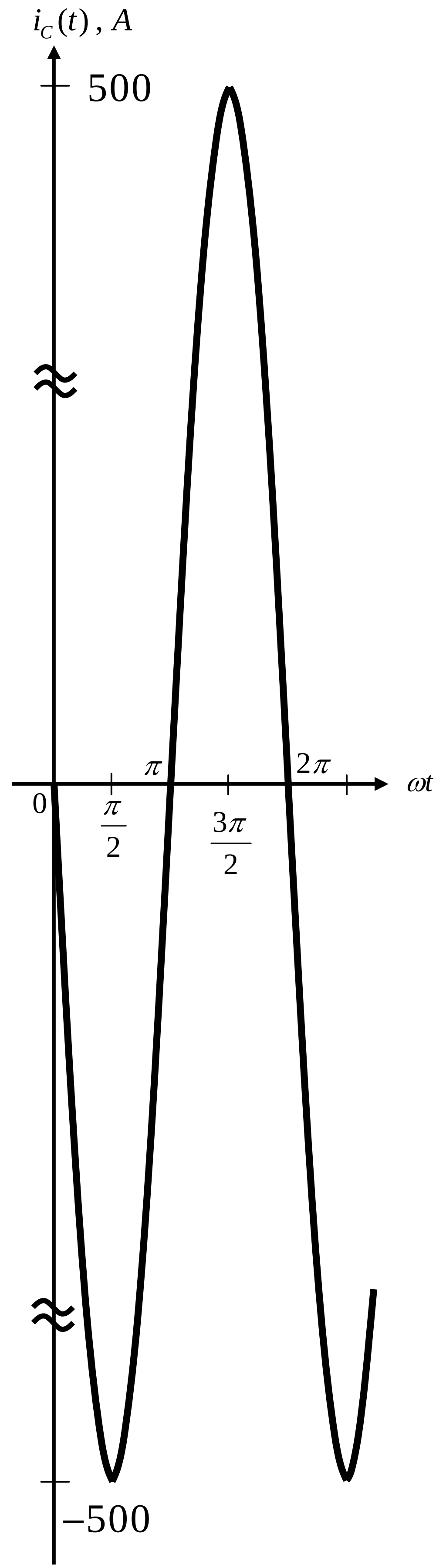
$$V(j2) = Z(j2)I(j2) = 250V$$

$$I_R(j2) = Y_R(j2)V(j2) = 1\,A$$

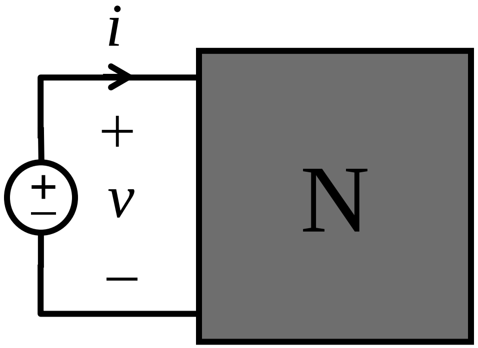
$$I_L(j2) = Y_L(j2)V(j2) = 500\angle -90^\circ\ A$$

$$I_C(j2) = Y_C(j2)V(j2) = 500\angle 90^\circ\ A$$





Instantaneous, Average, Complex Power



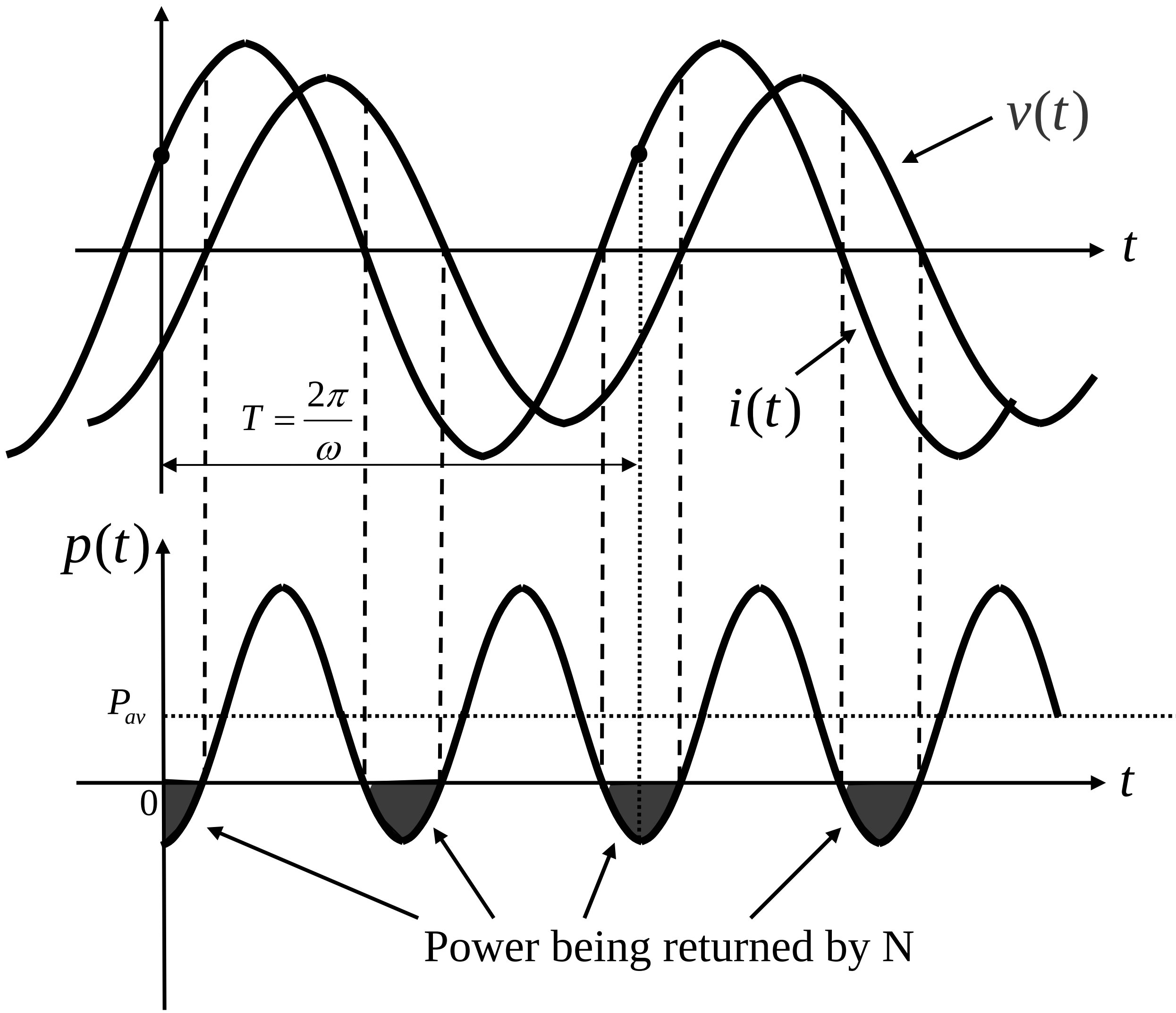
$$v(t) = |V|\cos\left(\omega t + \measuredangle V\right)$$

$$i(t) = |I|\cos\left(\omega t + \measuredangle I\right)$$

Instantaneous Power

$$p(t) = v(t)i(t) = |V||I|\cos(\omega t + \measuredangle V)\cos(\omega t + \measuredangle I)$$

$$= \underbrace{\frac{1}{2}|V||I|\cos(\measuredangle V - \measuredangle I)}_{\text{constant}} + \underbrace{\frac{1}{2}|V||I|\cos(2\omega t + \measuredangle V + \measuredangle I)}_{\text{Sinusoid of twice the frequency}}$$



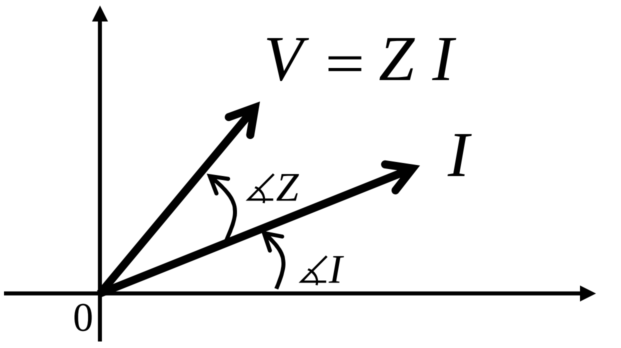
Average Power

$$P_{av} \triangleq \frac{1}{T}\int_0^T p(t)dt = \frac{1}{2}|V||I|\cos(\measuredangle V - \measuredangle I)$$

$$V = Z\, I$$

$$|V| e^{j\measuredangle V} = |Z| e^{j\measuredangle Z} \bullet |I| e^{j\measuredangle I}$$

$$= |Z||I| e^{j(\measuredangle Z + \measuredangle I)}$$



$$\therefore\ \measuredangle V = \measuredangle Z + \measuredangle I \quad \Rightarrow\ \measuredangle Z = \measuredangle V - \measuredangle I$$

Average Power $P_{av} = \frac{1}{2}|V||I|\cos \measuredangle Z$

Remarks:

1. $P_{av} = 0$ in $\begin{cases} \text{INDUCTOR} \\ \text{CAPACITOR} \end{cases}$

2. If N contains only passive elements, then

$$P_{av} \geq 0 \quad \Rightarrow \cos \measuredangle Z \geq 0$$

$$\therefore \quad -90^\circ \leq \measuredangle Z \leq 90^\circ \quad \text{for passive N}$$

Complex Power $P \triangleq \frac{1}{2} V \bar{I}$

$$P = \frac{1}{2}|V| e^{j\measuredangle V} \bullet |I| e^{-j\measuredangle I} = \frac{1}{2}|V||I| e^{j\underbrace{(\measuredangle V - \measuredangle I)}_{\measuredangle Z}}$$

$$P = \underbrace{\left(\frac{1}{2}|V||I|\cos\measuredangle Z\right)}_{P_{av} \triangleq \operatorname{Re} P} + j\underbrace{\left(\frac{1}{2}|V||I|\sin\measuredangle Z\right)}_{Q \triangleq \operatorname{Im} P}$$

Active power — Reactive power

$$\therefore \boxed{\begin{aligned} P_{av} &= \operatorname{Re}\left(\frac{1}{2} V \bar{I}\right) = \text{Average (Active) power} \\ Q &= \operatorname{Im}\left(\frac{1}{2} V \bar{I}\right) = \text{Reactive power} \end{aligned}} \quad \text{in Watts}$$

Effective (RMS) Value

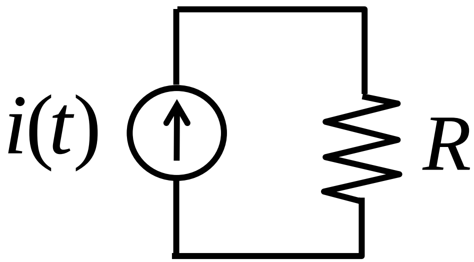
Definition : Given any **periodic** waveform $x(t)$ of period T,

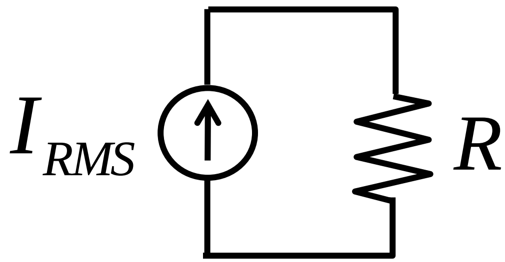
the ***RMS*** (root-mean-square) or effective value of $x(t)$

is defined as

$$X_{RMS} \triangleq \sqrt{\frac{1}{T}\int_{o}^{T} x^{2}(t)dt}$$

Interpretation





Let W_R = average power dissipated in Resistor

$$W_R = \frac{1}{T}\int_0^T \big(R\,i(t)\big)\big(i(t)\big)\,dt = R\underbrace{\left(\frac{1}{T}\int_0^T i^2(t)\,dt\right)}_{I_{RMS}^2} = R\,I_{RMS}^2$$

$\therefore$ The same average power is dissipated if the resistor is driven by a dc current source of value I_{RMS} ; hence I_{RMS} is called the effective value of $i(t)$.

Sinusoidal waveforms: $x(t) = \left|X\right| \cos(\omega t + \measuredangle X)$

$$X_{RMS} = \frac{\left|X\right|}{\sqrt{2}}$$

Note:

1. Since most instruments measure RMS values; hence our 60Hz-sinusoidal voltages are rated in RMS values.

   Example: Line voltage =110V $\Rightarrow$ magnitude=$\sqrt{2}(110)V$.

2. $P_{av} = \frac{\left|V\right|}{\sqrt{2}} \bullet \frac{\left|I\right|}{\sqrt{2}} \cos \measuredangle Z = V_{RMS} I_{RMS} \cos \measuredangle Z$

Significance of Complex Power

Most electrical machines are designed to withstand a maximum voltage magnitude $|V|$ and a maximum current magnitude $|I|$. Hence, electrical machines are rated in maximum $|P| = \frac{1}{2}|V||I|$ in KVA, and not in maximum average power dissipation P_{av}.

Power factor $PF \triangleq \dfrac{P_{av}}{\frac{1}{2}|V||I|} = \dfrac{|V||I|\cos\measuredangle Z}{|V||I|}$

$$\therefore \quad \boxed{PF = \cos\measuredangle Z}$$

$$PF = \begin{cases} 1 & \text{for Resistors} \\ 0 & \text{for INDUCTORS and CAPACITORS} \end{cases}$$

Note:

1. For power generation companies, it is important to keep the *PF* of the load (customers) be as close to unity as possible.

2. The ***Watthour meter*** measures P_{av}, **not** $|P|$.

   Example: If a factory dissipates 10*KW* of power with 50 % *PF*, then the power company must generate 20 *KVA* of power.

$\therefore$ A penalty is usually levied for low *PF* customers.