$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} & a_{17} & a_{18} & a_{19} & a_{1,10} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} & a_{27} & a_{28} & a_{29} & a_{2,10} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} & a_{37} & a_{38} & a_{39} & a_{3,10} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} & a_{47} & a_{48} & a_{49} & a_{4,10} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & a_{56} & a_{57} & a_{58} & a_{59} & a_{5,10} \\ a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & a_{66} & a_{67} & a_{68} & a_{69} & a_{6,10} \\ a_{71} & a_{72} & a_{73} & a_{74} & a_{75} & a_{76} & a_{77} & a_{78} & a_{79} & a_{7,10} \\ a_{81} & a_{82} & a_{83} & a_{84} & a_{85} & a_{86} & a_{87} & a_{88} & a_{89} & a_{8,10} \\ a_{91} & a_{92} & a_{93} & a_{94} & a_{95} & a_{96} & a_{97} & a_{98} & a_{99} & a_{9,10} \\ a_{10,1} & a_{10,2} & a_{10,3} & a_{10,4} & a_{10,5} & a_{10,6} & a_{10,7} & a_{10,8} & a_{10,9} & a_{10,10} \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \\ x_9 \\ x_{10} \end{bmatrix}}_{\mathbf{x}} = \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \\ b_6 \\ b_7 \\ b_8 \\ b_9 \\ b_{10} \end{bmatrix}}_{\mathbf{b}}$$

CRAMER’S RULE

If $\det \mathbf{A} \neq 0$, then the solution $\mathbf{x} = \left[x_i \right]$ of the system of linear equations

$$\boxed{\mathbf{Ax} = \mathbf{b}}$$

is given by

$$\boxed{x_i = \frac{\Delta_i}{\Delta}}$$

$$i = 1, 2, \cdots, n$$

where $\Delta = \det \mathbf{A}$ and $\Delta_i = \det \mathbf{A}_{(i)}$, where $\mathbf{A}_{(i)}$ is the matrix obtained by replacing the *i*th column of **A** by **b**.

Determinant of A

For a 3 x 3 matrix

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\begin{aligned} \det \mathbf{A} = {} & a_{11}(a_{22}a_{33} - a_{23}a_{32}) \\ & -a_{21}(a_{12}a_{33} - a_{13}a_{32}) \\ & +a_{31}(a_{12}a_{23} - a_{22}a_{13}) \triangleq \Delta \end{aligned}$$

Remark

An extension of the above procedure can be used to find det **A** of any *n* x *n* matrix **M**. However, for *n* > 3, it is too cumbersome and error prone to compute **determinants** manually. Almost every numerical analysis software includes a subroutine to compute determinants.

Determinant of $\mathbf{A}$

Given

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

$$\det \mathbf{A} = \sum_{j=1}^{n} a_{jk} A_{jk}$$

for any $\quad k = 1, 2, \cdots, n$

Minor and Cofactor of a_{ij} of **A**

Definition:

The determinant of the (n-1) x (n-1) matrix formed by deleting the ith row and the jth column of the n x n matrix **A** is called the **minor** of element a_{ij} of **A**, and is denoted by M_{ij}.

The number

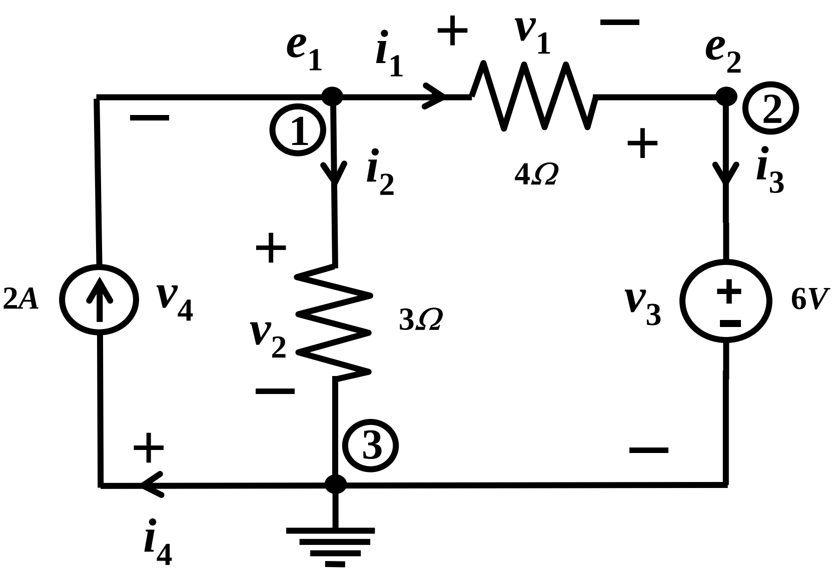
$$A_{ij} = (-1)^{i+j} M_{ij}$$

is called the **cofactor** of a_{ij}.

Remark

The signs $(-1)^{i+j}$ form a checkerboard pattern:

$$\begin{bmatrix} + & - & + & \cdots \\ - & + & - & \cdots \\ + & - & + & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

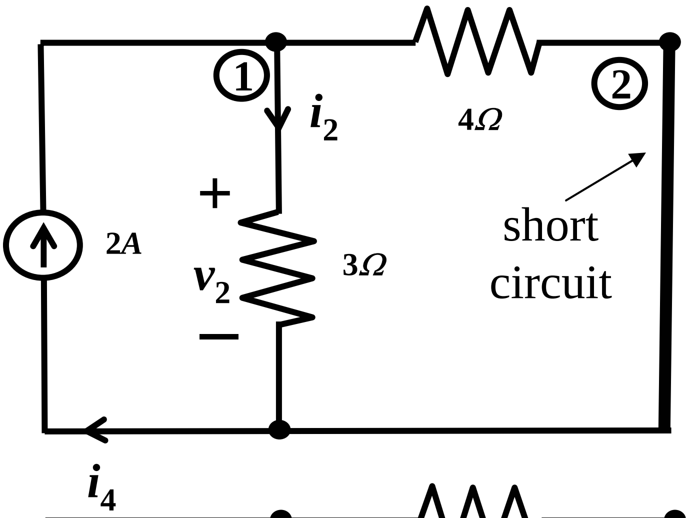


Problem :

Find v_2.

Solution-by-Inspection

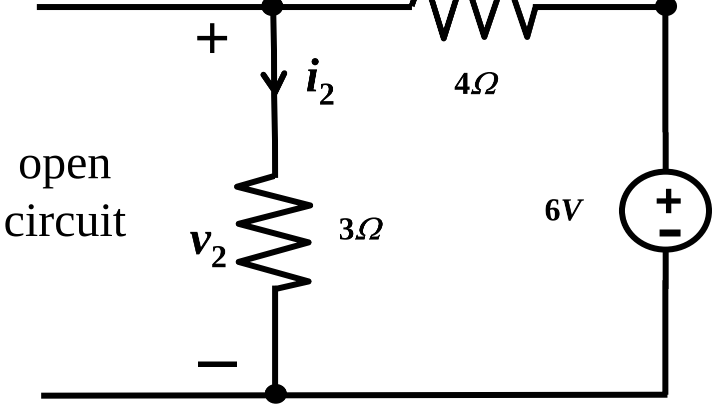
Step1. Find voltage component solutions.



The 2 resistors in the simplified circuit are connected in parallel:

$$v_2' = \left(\frac{3 \bullet 4}{3+4}\right)(2)$$

$$= \frac{24}{7}V$$



The simplified circuit is a voltage divider:

$$v_2'' = \left(\frac{3}{3+4}\right)(6)$$

$$= \frac{18}{7}V$$

Step2. Add voltage component solutions.

$$v_2 = v_1' + v_2'' = \frac{24}{7} + \frac{18}{7} = \frac{42}{7}V$$

Let us rearrange all 10 independent equations as follow:

$$\left[\begin{array}{cc|cccc|cccc}
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\
\hline
-1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & -1 & 0 & 0 & 0 & 4 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 3 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}\right]
\left[\begin{array}{c}
e_1 \\ e_2 \\ \hline v_1 \\ v_2 \\ v_3 \\ v_4 \\ \hline i_1 \\ i_2 \\ i_3 \\ i_4
\end{array}\right]
=
\left[\begin{array}{c}
0 \\ 0 \\ \hline 0 \\ 0 \\ 0 \\ 0 \\ \hline 0 \\ 0 \\ 6 \\ 2
\end{array}\right]$$

$$\det \mathbf{A} = \underbrace{\left|\begin{array}{cc:cccc:cccc}
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\
\hdashline
-1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
\hdashline
0 & 0 & -1 & 0 & 0 & 0 & 4 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 3 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}\right|}_{\mathbf{A}} \triangleq \Delta$$

$$\det \mathbf{A}_1 = \left| \underbrace{\begin{array}{cc:cccc:cccc}
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\
\hdashline
0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
\hdashline
0 & 0 & -1 & 0 & 0 & 0 & 4 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 3 & 0 & 0 \\
6 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}}_{\mathbf{A}_1} \right| \triangleq \Delta_1$$

$$\det \mathbf{A}_4 = \underbrace{\left|\begin{array}{cc:cccc:cccc}
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\
\hdashline
-1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
\hdashline
0 & 0 & -1 & 0 & 0 & 0 & 4 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 \\
0 & 0 & 0 & 6 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}\right|}_{\mathbf{A}_4} \triangleq \Delta_4$$

$$\det \mathbf{A}_9 = \left| \underbrace{\begin{array}{cc:cccc:cccc}
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
\hdashline
-1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
\hdashline
0 & 0 & -1 & 0 & 0 & 0 & 4 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 3 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 6 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 1
\end{array}}_{\mathbf{A}_9} \right| \triangleq \Delta_9$$

$$e_1 = \frac{\Delta_1}{\Delta}$$

$$= \frac{1}{\Delta}\left\{\begin{array}{c} 0 \bullet A_{11} + 0 \bullet A_{21} + 0 \bullet A_{31} + 0 \bullet A_{41} + 0 \bullet A_{51} \\ + 0 \bullet A_{61} + 0 \bullet A_{71} + 0 \bullet A_{81} + 6 \bullet A_{91} + 2 \bullet A_{10,1} \end{array}\right\}$$

$$= \frac{1}{\Delta}\left(6 \bullet A_{91} + 2 \bullet A_{10,1}\right), \quad \text{because } a_{11} = a_{21} = \cdots a_{81} = 0$$

$$= \underbrace{\frac{A_{91}}{\Delta}}_{k_{11}}(6) + \underbrace{\frac{A_{10,1}}{\Delta}}_{k_{12}}(2)$$

$$= k_{11} \bullet 6 + k_{12} \bullet 2$$

$$= k_{11} \bullet v_{s1} + k_{12} \bullet i_{s1}$$

$$i_9 = \frac{\Delta_9}{\Delta}$$

$$= \frac{1}{\Delta}\left\{\begin{array}{l} 0 \bullet A_{19} + 0 \bullet A_{29} + 0 \bullet A_{39} + 0 \bullet A_{49} + 0 \bullet A_{59} \\ +0 \bullet A_{69} + 0 \bullet A_{79} + 0 \bullet A_{89} + 6 \bullet A_{99} + 2 \bullet A_{10,9} \end{array}\right\}$$

$$= \frac{1}{\Delta}\left(6 \bullet A_{99} + 2 \bullet A_{10,9}\right)$$

$$= \underbrace{\frac{A_{99}}{\Delta}}_{k''_{11}}(6) + \underbrace{\frac{A_{10,9}}{\Delta}}_{k''_{12}}(2)$$

$$= k''_{11} \bullet v_{s1} + k''_{12} \bullet i_{s1}$$

$$v_2 = \frac{\Delta_4}{\Delta}$$

$$= \frac{1}{\Delta}\left\{ \begin{array}{c} 0\bullet A_{14} + 0\bullet A_{24} + 0\bullet A_{34} + 0\bullet A_{44} + 0\bullet A_{54} \\ +0\bullet A_{64} + 0\bullet A_{74} + 0\bullet A_{84} + 6\bullet A_{94} + 2\bullet A_{10,4} \end{array} \right\}$$

$$= \frac{1}{\Delta}\left(6\bullet A_{94} + 2\bullet A_{10,4}\right)$$

$$= \underbrace{\frac{A_{94}}{\Delta}}_{k'_{11}}(6) + \underbrace{\frac{A_{10,4}}{\Delta}}_{k'_{12}}(2)$$

$$= k'_{11} \bullet v_{s1} + k'_{12} \bullet i_{s1}$$

Let us rearrange all 10 independent equations as follow:

$$
\left[\begin{array}{cc|cccc|cccc}
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\
\hline
-1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & -1 & 0 & 0 & 0 & 4\frac{d}{dt} & 0 & 0 & 0 \\
0 & 0 & 0 & 2\frac{d}{dt} & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}\right]
\left[\begin{array}{c}
e_1 \\ e_2 \\ \hline v_1 \\ v_2 \\ v_3 \\ v_4 \\ \hline i_1 \\ i_2 \\ i_3 \\ i_4
\end{array}\right]
=
\left[\begin{array}{c}
0 \\ 0 \\ \hline 0 \\ 0 \\ 0 \\ 0 \\ \hline 0 \\ 0 \\ 6\sin\omega t \\ 2\cos\omega t
\end{array}\right]
$$

Let us rearrange all 10 independent equations as follow:

$$\left[\begin{array}{cc|cccc|cccc}
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\
\hline
-1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & -1 & 0 & 0 & 0 & L_1 \frac{d}{dt} & 0 & 0 & 0 \\
0 & 0 & 0 & C_2 \frac{d}{dt} & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}\right]
\left[\begin{array}{c}
e_1 \\ e_2 \\ \hline v_1 \\ v_2 \\ v_3 \\ v_4 \\ \hline i_1 \\ i_2 \\ i_3 \\ i_4
\end{array}\right]
=
\left[\begin{array}{c}
0 \\ 0 \\ \hline 0 \\ 0 \\ 0 \\ 0 \\ \hline 0 \\ 0 \\ 6\sin\omega t \\ 2\cos\omega t
\end{array}\right]$$

$$\underbrace{\left[\begin{array}{cccc} -1 & 0 & 0 & 0 \\ 0 & 2\frac{d}{dt} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right|}_{\mathbf{H}_v}\underbrace{\left.\begin{array}{cccc} 4\frac{d}{dt} & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]}_{\mathbf{H}_i}\underbrace{\left[\begin{array}{c} v_1 \\ v_2 \\ v_3 \\ v_4 \\ \hline i_1 \\ i_2 \\ i_3 \\ i_4 \end{array}\right]}_{\begin{bmatrix}\mathbf{v} \\ \mathbf{i}\end{bmatrix}} = \underbrace{\begin{bmatrix} 0 \\ 0 \\ 6\sin\omega t \\ 2\cos\omega t \end{bmatrix}}_{\substack{\mathbf{u} \\ \uparrow \\ \text{independent} \\ \text{source} \\ \text{vector}}}$$

$$\Downarrow$$

$$\boxed{\mathbf{H}_v\,\mathbf{v} + \mathbf{H}_i\,\mathbf{i} = \mathbf{u}}$$

$$\left[\underbrace{\begin{matrix} -1 & 0 & 0 & 0 \\ 0 & C_2\frac{d}{dt} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{matrix}}_{\mathbf{H}_v} \;\middle|\; \underbrace{\begin{matrix} L_1\frac{d}{dt} & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{matrix}}_{\mathbf{H}_i}\right] \underbrace{\left[\begin{array}{c} v_1 \\ v_2 \\ v_3 \\ v_4 \\ \hline i_1 \\ i_2 \\ i_3 \\ i_4 \end{array}\right]}_{\begin{bmatrix}\mathbf{v} \\ \mathbf{i}\end{bmatrix}} = \underbrace{\begin{bmatrix} 0 \\ 0 \\ 6\sin\omega t \\ 2\cos\omega t \end{bmatrix}}_{\mathbf{u}}$$

independent
source
vector

$$\Downarrow$$

$$\boxed{\mathbf{H}_v\,\mathbf{v} + \mathbf{H}_i\,\mathbf{i} = \mathbf{u}}$$