Stable Marriage Problem

• Small town with \( n \) boys and \( n \) girls.
• Each girl has a ranked preference list of boys.
• Each boy has a ranked preference list of girls.

How should they be matched?
Stable Marriage Problem

• Small town with $n$ boys and $n$ girls.
Stable Marriage Problem

• Small town with \( n \) boys and \( n \) girls.
• Each girl has a ranked preference list of boys.
Stable Marriage Problem

- Small town with $n$ boys and $n$ girls.
- Each girl has a ranked preference list of boys.
- Each boy has a ranked preference list of girls.
Stable Marriage Problem

• Small town with $n$ boys and $n$ girls.
• Each girl has a ranked preference list of boys.
• Each boy has a ranked preference list of girls.

How should they be matched?
Count the ways..

• Maximize total satisfaction.
Count the ways..

- Maximize total satisfaction.
- Maximize number of first choices.
Count the ways..

• Maximize total satisfaction.
• Maximize number of first choices.
• Maximize worse off.
Count the ways..

- Maximize total satisfaction.
- Maximize number of first choices.
- Maximize worse off.
- Minimize difference between preference ranks.
The best laid plans..

Consider the couples..

- Jennifer and Brad
- Angelina and Billy-Bob
The best laid plans..

Consider the couples..

• Jennifer and Brad
• Angelina and Billy-Bob

Brad prefers Angelina to Jennifer.
The best laid plans..

Consider the couples..

• Jennifer and Brad
• Angelina and Billy-Bob

Brad prefers Angelina to Jennifer.

Angelina prefers Brad to BillyBob.
Consider the couples..

- Jennifer and Brad
- Angelina and Billy-Bob

Brad prefers Angelina to Jennifer.

Angelina prefers Brad to BillyBob.

Uh..oh.
So..

Produce a pairing where there is no running off!
Produce a pairing where there is no running off!

**Definition:** A **pairing** is disjoint set of \( n \) boy-girl pairs.
Produce a pairing where there is no running off!

**Definition:** A *pairing* is disjoint set of $n$ boy-girl pairs.

Example: A pairing $S = \{(Brad, Jen); (BillyBob, Angelina)\}$. 
Produce a pairing where there is no running off!

**Definition:** A **pairing** is disjoint set of \( n \) boy-girl pairs.

Example: A pairing \( S = \{(Brad, Jen); (BillyBob, Angelina)\} \).

**Definition:** A **rogue couple** \( b, g^* \) for a pairing \( S \): 
\( b \) and \( g^* \) prefer each other to their partners in \( S \).
Produce a pairing where there is no running off!

**Definition:** A *pairing* is disjoint set of $n$ boy-girl pairs.

Example: A pairing $S = \{(Brad, Jen); (BillyBob, Angelina)\}$.

**Definition:** A *rogue couple* $b, g^*$ for a pairing $S$: $b$ and $g^*$ prefer each other to their partners in $S$.

Example: Brad and Angelina are a rogue couple in $S$. 
A stable pairing??

Given a set of preferences.
A stable pairing??

Given a set of preferences.

Is there a stable pairing?

How does one find it?
Given a set of preferences.

Is there a stable pairing?

How does one find it?

Consider a single gender version: stable roommates.

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>B</td>
<td>C</td>
<td>D</td>
</tr>
<tr>
<td>B</td>
<td>C</td>
<td>A</td>
<td>D</td>
</tr>
<tr>
<td>C</td>
<td>A</td>
<td>B</td>
<td>D</td>
</tr>
<tr>
<td>D</td>
<td>A</td>
<td>B</td>
<td>C</td>
</tr>
</tbody>
</table>

A stable pairing??
A stable pairing??

Given a set of preferences.

Is there a stable pairing?

How does one find it?

Consider a single gender version: stable roommates.

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
</tr>
</thead>
<tbody>
<tr>
<td>B</td>
<td>C</td>
<td>A</td>
<td>D</td>
</tr>
<tr>
<td>C</td>
<td>A</td>
<td>B</td>
<td>D</td>
</tr>
<tr>
<td>D</td>
<td>A</td>
<td>B</td>
<td>C</td>
</tr>
</tbody>
</table>

A

B

C

D

5
A stable pairing??

Given a set of preferences.

Is there a stable pairing?

How does one find it?

Consider a single gender version: stable roommates.

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td></td>
<td>B</td>
<td>C</td>
<td>D</td>
</tr>
<tr>
<td>B</td>
<td></td>
<td>C</td>
<td>A</td>
<td>D</td>
</tr>
<tr>
<td>C</td>
<td></td>
<td>A</td>
<td>B</td>
<td>D</td>
</tr>
<tr>
<td>D</td>
<td></td>
<td>A</td>
<td>B</td>
<td>C</td>
</tr>
</tbody>
</table>

\[\text{Diagram of preferences:}
\]

A ➔ B ➔ C ➔ D

B ➔ C ➔ A ➔ D

C ➔ A ➔ B ➔ D

D ➔ A ➔ B ➔ C

\]

Is there a stable pairing?

How does one find it?
A stable pairing??

Given a set of preferences.

Is there a stable pairing?

How does one find it?

Consider a single gender version: stable roommates.

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>B</td>
<td>C</td>
<td>D</td>
<td></td>
</tr>
<tr>
<td>B</td>
<td>C</td>
<td>A</td>
<td>D</td>
<td></td>
</tr>
<tr>
<td>C</td>
<td>A</td>
<td>B</td>
<td>D</td>
<td></td>
</tr>
<tr>
<td>D</td>
<td>A</td>
<td>B</td>
<td>C</td>
<td></td>
</tr>
</tbody>
</table>

A diagram is shown with nodes A, B, C, and D, illustrating a stable pairing.
Given a set of preferences.

Is there a stable pairing?

How does one find it?

Consider a single gender version: stable roommates.

\[
\begin{array}{c|cccc}
A & B & C & D \\
B & C & A & D \\
C & A & B & D \\
D & A & B & C \\
\end{array}
\]
A stable pairing??

Given a set of preferences.

Is there a stable pairing?

How does one find it?

Consider a single gender version: stable roommates.

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td></td>
<td>B</td>
<td>C</td>
<td>D</td>
</tr>
<tr>
<td>B</td>
<td>C</td>
<td>A</td>
<td>D</td>
<td></td>
</tr>
<tr>
<td>C</td>
<td>A</td>
<td>B</td>
<td>D</td>
<td></td>
</tr>
<tr>
<td>D</td>
<td>A</td>
<td>B</td>
<td>C</td>
<td></td>
</tr>
</tbody>
</table>

\[ \begin{align*}
A & \rightarrow B & C & \rightarrow D \\
B & \rightarrow C & A & \rightarrow D \\
C & \rightarrow A & B & \rightarrow D \\
D & \rightarrow A & B & \rightarrow C \\
\end{align*} \]
A stable pairing??

Given a set of preferences.

Is there a stable pairing?

How does one find it?

Consider a single gender version: stable roommates.

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>B</td>
<td>C</td>
<td>D</td>
<td></td>
</tr>
<tr>
<td>B</td>
<td>C</td>
<td>A</td>
<td>D</td>
<td></td>
</tr>
<tr>
<td>C</td>
<td>A</td>
<td>B</td>
<td>D</td>
<td></td>
</tr>
<tr>
<td>D</td>
<td>A</td>
<td>B</td>
<td>C</td>
<td></td>
</tr>
</tbody>
</table>

A – B

C – D
A stable pairing??

Given a set of preferences.

Is there a stable pairing?

How does one find it?

Consider a single gender version: stable roommates.

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>B</td>
<td>C</td>
<td>D</td>
<td></td>
</tr>
<tr>
<td>B</td>
<td>C</td>
<td>A</td>
<td>D</td>
<td></td>
</tr>
<tr>
<td>C</td>
<td>A</td>
<td>B</td>
<td>D</td>
<td></td>
</tr>
<tr>
<td>D</td>
<td>A</td>
<td>B</td>
<td>C</td>
<td></td>
</tr>
</tbody>
</table>
A stable pairing??

Given a set of preferences.

Is there a stable pairing?

How does one find it?

Consider a single gender version: stable roommates.

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>B</td>
<td>C</td>
<td>D</td>
<td></td>
</tr>
<tr>
<td>B</td>
<td>C</td>
<td>A</td>
<td>D</td>
<td></td>
</tr>
<tr>
<td>C</td>
<td>A</td>
<td>B</td>
<td>D</td>
<td></td>
</tr>
<tr>
<td>D</td>
<td>A</td>
<td>B</td>
<td>C</td>
<td></td>
</tr>
</tbody>
</table>

A stable pairing is found when no pair prefers the other to their current partner. The diagram above illustrates a stable pairing where no tomboy (A or D) would prefer to change partners.
The Traditional Marriage Algorithm.

Each Day:
1. Each boy proposes to his favorite girl on his list.
2. Each girl rejects all but her favorite proposer (whom she puts on a string.)
3. Rejected boy crosses rejecting girl off his list.

Stop when each girl gets exactly one proposal.

Does this terminate?

...produce a pairing?

...a stable pairing?

Do boys or girls do “better”? 
The Traditional Marriage Algorithm.

Each Day:

1. Each boy proposes to his favorite girl on his list.
2. Each girl rejects all but her favorite proposer (whom she puts on a string).
3. Rejected boy crosses rejecting girl off his list.

Stop when each girl gets exactly one proposal.

Does this terminate? 

...produce a pairing? 

....a stable pairing? 

Do boys or girls do "better"?
The Traditional Marriage Algorithm.

Each Day:

1. Each boy proposes to his favorite girl on his list.
The Traditional Marriage Algorithm.

Each Day:

1. Each boy proposes to his favorite girl on his list.
2. Each girl rejects all but her favorite proposer (whom she puts on a string.)
The Traditional Marriage Algorithm.

Each Day:

1. Each boy **proposes** to his favorite girl on his list.
2. Each girl rejects all but her favorite proposer (whom she puts on a **string**.)
3. Rejected boy **crosses** rejecting girl off his list.
The Traditional Marriage Algorithm.

Each Day:

1. Each boy *proposes* to his favorite girl on his list.

2. Each girl rejects all but her favorite proposer (whom she puts on a *string*.)

3. Rejected boy *crosses* rejecting girl off his list.

Stop when each girl gets exactly one proposal.
The Traditional Marriage Algorithm.

Each Day:

1. Each boy **proposes** to his favorite girl on his list.
2. Each girl rejects all but her favorite proposer (whom she puts on a string.)
3. Rejected boy **crosses** rejecting girl off his list.

Stop when each girl gets exactly one proposal.

Does this terminate?
The Traditional Marriage Algorithm.

Each Day:

1. Each boy \textbf{proposes} to his favorite girl on his list.
2. Each girl rejects all but her favorite proposer (whom she puts on a \textit{string}).
3. Rejected boy \textit{crosses} rejecting girl off his list.

Stop when each girl gets exactly one proposal.

Does this terminate?

...produce a pairing?
The Traditional Marriage Algorithm.

Each Day:

1. Each boy *proposes* to his favorite girl on his list.
2. Each girl rejects all but her favorite proposer (whom she puts on a *string*.)
3. Rejected boy *crosses* rejecting girl off his list.

Stop when each girl gets exactly one proposal.

Does this terminate?

...produce a pairing?

....a stable pairing?
The Traditional Marriage Algorithm.

Each Day:

1. Each boy **proposes** to his favorite girl on his list.
2. Each girl rejects all but her favorite proposer (whom she puts on a **string**.)
3. Rejected boy **crosses** rejecting girl off his list.

Stop when each girl gets exactly one proposal.

Does this terminate?

...produce a pairing?

....a stable pairing?

Do boys or girls do “better”? 
The Traditional Marriage Algorithm.

Each Day:

1. Each boy \textbf{proposes} to his favorite girl on his list.
2. Each girl rejects all but her favorite proposer (whom she puts on a \textit{string}.)
3. Rejected boy \textit{crosses} rejecting girl off his list.

Stop when each girl gets exactly one proposal.
Does this terminate?

...produce a pairing?

....a stable pairing?

Do boys or girls do “better”? 
### Example.

<table>
<thead>
<tr>
<th>Boys</th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>B</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>C</td>
<td>2</td>
<td>1</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Girls</th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>C</td>
<td>A</td>
</tr>
<tr>
<td>2</td>
<td>A</td>
<td>B</td>
</tr>
<tr>
<td>3</td>
<td>A</td>
<td>C</td>
</tr>
</tbody>
</table>

Day 1

Day 2

Day 3

Day 4

Day 5
Example.

<table>
<thead>
<tr>
<th>Boys</th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>B</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>C</td>
<td>2</td>
<td>1</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Girls</th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>C</td>
<td>A</td>
</tr>
<tr>
<td>2</td>
<td>A</td>
<td>B</td>
</tr>
<tr>
<td>3</td>
<td>A</td>
<td>C</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Day 1</th>
<th>Day 2</th>
<th>Day 3</th>
<th>Day 4</th>
<th>Day 5</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Example.

<table>
<thead>
<tr>
<th>Boys</th>
<th></th>
<th>Girls</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>1 2 3</td>
<td>1</td>
<td>C A B</td>
</tr>
<tr>
<td>B</td>
<td>1 2 3</td>
<td>2</td>
<td>A B C</td>
</tr>
<tr>
<td>C</td>
<td>2 1 3</td>
<td>3</td>
<td>A C B</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Day 1</th>
<th>Day 2</th>
<th>Day 3</th>
<th>Day 4</th>
<th>Day 5</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>A, B</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
<td>C</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Example.

<table>
<thead>
<tr>
<th></th>
<th>Boys</th>
<th></th>
<th>Girls</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>A</td>
<td>1 2 3</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>B</td>
<td>X 2 3</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>C</td>
<td>2 1 3</td>
<td>3</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>Day 1</th>
<th>Day 2</th>
<th>Day 3</th>
<th>Day 4</th>
<th>Day 5</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>A B</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
<td>C</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Example.

<table>
<thead>
<tr>
<th>Boys</th>
<th>Girls</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>1</td>
</tr>
<tr>
<td>B</td>
<td>X</td>
</tr>
<tr>
<td>C</td>
<td>2</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Day 1</th>
<th>Day 2</th>
<th>Day 3</th>
<th>Day 4</th>
<th>Day 5</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>A, B</td>
<td>A</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>C</td>
<td>B, C</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Example.

<table>
<thead>
<tr>
<th>Boys</th>
<th>Girls</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>1</td>
</tr>
<tr>
<td>B</td>
<td>x1</td>
</tr>
<tr>
<td>x2</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Day 1</th>
<th>Day 2</th>
<th>Day 3</th>
<th>Day 4</th>
<th>Day 5</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>A, B</td>
<td>A</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>C</td>
<td>B, C</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
### Example.

<table>
<thead>
<tr>
<th>Boys</th>
<th></th>
<th>Girls</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>1 2 3</td>
<td>1</td>
<td>C A B</td>
</tr>
<tr>
<td>B</td>
<td>X 2 3</td>
<td>2</td>
<td>A B C</td>
</tr>
<tr>
<td>C</td>
<td>X 1 3</td>
<td>3</td>
<td>A C B</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Day 1</th>
<th>Day 2</th>
<th>Day 3</th>
<th>Day 4</th>
<th>Day 5</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>A, B</td>
<td>A</td>
<td>A, C</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>C</td>
<td>B, C</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
<td></td>
<td>B</td>
<td></td>
</tr>
</tbody>
</table>
Example.

<table>
<thead>
<tr>
<th></th>
<th>Boys</th>
<th></th>
<th>Girls</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>A</td>
<td>X</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>B</td>
<td>X</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>C</td>
<td>X</td>
<td>2</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>Day 1</th>
<th>Day 2</th>
<th>Day 3</th>
<th>Day 4</th>
<th>Day 5</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>A, B</td>
<td>A</td>
<td>A, C</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>C</td>
<td>B, C</td>
<td>B</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Example.

<table>
<thead>
<tr>
<th>Boys</th>
<th></th>
<th>Girls</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>X</td>
<td>1</td>
</tr>
<tr>
<td>B</td>
<td>X</td>
<td>2</td>
</tr>
<tr>
<td>C</td>
<td>X</td>
<td>3</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Day 1</th>
<th>Day 2</th>
<th>Day 3</th>
<th>Day 4</th>
<th>Day 5</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>A, B</td>
<td>A</td>
<td>A, C</td>
<td>C</td>
</tr>
<tr>
<td>2</td>
<td>C</td>
<td>B, C</td>
<td>B</td>
<td>A, B</td>
</tr>
<tr>
<td>3</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Example.

<table>
<thead>
<tr>
<th>Boys</th>
<th>Girls</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>1</td>
</tr>
<tr>
<td>B</td>
<td>2</td>
</tr>
<tr>
<td>C</td>
<td>3</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>A</th>
<th>1</th>
<th>C</th>
<th>A</th>
<th>B</th>
</tr>
</thead>
<tbody>
<tr>
<td>X</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>X</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>X</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Day 1</th>
<th>Day 2</th>
<th>Day 3</th>
<th>Day 4</th>
<th>Day 5</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>A, B</td>
<td>A</td>
<td>A, C</td>
<td>C</td>
</tr>
<tr>
<td>2</td>
<td>C</td>
<td>B, C</td>
<td>B</td>
<td>A, B</td>
</tr>
<tr>
<td>3</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Example.

<table>
<thead>
<tr>
<th>Boys</th>
<th>Girls</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>1</td>
</tr>
<tr>
<td>X</td>
<td>C</td>
</tr>
<tr>
<td>B</td>
<td>X</td>
</tr>
<tr>
<td>X</td>
<td>A</td>
</tr>
<tr>
<td>C</td>
<td>X</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>1</td>
<td>A</td>
</tr>
<tr>
<td>2</td>
<td>B</td>
</tr>
<tr>
<td>3</td>
<td>C</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Day 1</th>
<th>Day 2</th>
<th>Day 3</th>
<th>Day 4</th>
<th>Day 5</th>
</tr>
</thead>
<tbody>
<tr>
<td>A, B</td>
<td>A</td>
<td>A, C</td>
<td>C</td>
<td>C</td>
</tr>
<tr>
<td>C</td>
<td>B, C</td>
<td>B</td>
<td>A, B</td>
<td>A</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Example.

<table>
<thead>
<tr>
<th></th>
<th>Boys</th>
<th></th>
<th></th>
<th></th>
<th></th>
<th>Girls</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>X 1</td>
<td>2</td>
<td>3</td>
<td></td>
<td>C</td>
<td>A</td>
</tr>
<tr>
<td>B</td>
<td>X 1</td>
<td>X</td>
<td>2</td>
<td></td>
<td>A</td>
<td>B</td>
</tr>
<tr>
<td>C</td>
<td>X 2</td>
<td>1</td>
<td>3</td>
<td></td>
<td>A</td>
<td>C</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>Day 1</th>
<th></th>
<th>Day 2</th>
<th></th>
<th>Day 3</th>
<th></th>
<th>Day 4</th>
<th></th>
<th>Day 5</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>A, B</td>
<td>X</td>
<td>A</td>
<td>X</td>
<td>A, C</td>
<td>X</td>
<td>C</td>
<td>C</td>
<td>C</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>C</td>
<td></td>
<td>B, C</td>
<td></td>
<td>B</td>
<td></td>
<td>A, B</td>
<td>A</td>
<td>A</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>B</td>
</tr>
</tbody>
</table>
Termination.

Every non-terminated day a boy crossed an item off the list. Total size of lists? $n$ boys, $n$ length list. Terminates in at most $n^2 + 1$ steps!
Every non-terminated day a boy crossed an item off the list.
Every non-terminated day a boy crossed an item off the list.

Total size of lists?
Termination.

Every non-terminated day a boy crossed an item off the list.

Total size of lists? $n$ boys, $n$ length list.
Every non-terminated day a boy crossed an item off the list.

Total size of lists? $n$ boys, $n$ length list. $n^2$
Every non-terminated day a boy crossed an item off the list.

Total size of lists? $n$ boys, $n$ length list. $n^2$

Terminates in at most $n^2 + 1$ steps!
It gets better every day for girls..
It gets better every day for girls.

Improvement Lemma: It just gets better for girls.
Improvement Lemma: It just gets better for girls.
If on day $t$ a girl, $g$, has a boy $b$ on a string,
**Improvement Lemma:** It just gets better for girls.

If on day $t$ a girl, $g$, has a boy $b$ on a string, any boy, $b'$, on $g$'s string for any day $t' > t$
Improvement Lemma: It just gets better for girls.
If on day $t$ a girl, $g$, has a boy $b$ on a string, any boy, $b'$, on $g$’s string for any day $t' > t$ is at least as good as $b$. 
Improvement Lemma: It just gets better for girls.
If on day $t$ a girl, $g$, has a boy $b$ on a string, any boy, $b'$, on $g$'s string for any day $t' > t$ is at least as good as $b$.

Proof:
Improvement Lemma: It just gets better for girls.
If on day $t$ a girl, $g$, has a boy $b$ on a string, any boy, $b'$, on $g$’s string for any day $t' > t$ is at least as good as $b$.

Proof:
$P(k)$- "boy on $g$’s string is at least as good as $b$ on day $t + k"
**Improvement Lemma**: It just gets better for girls.

If on day $t$ a girl, $g$, has a boy $b$ on a string, any boy, $b'$, on $g$’s string for any day $t' > t$ is at least as good as $b$.

**Proof:**

$P(k)$ - “boy on $g$’s string is at least as good as $b$ on day $t + k$”

$P(0)$ – true. Girl has $b$ on string.
**Improvement Lemma: It just gets better for girls.**

If on day $t$ a girl, $g$, has a boy $b$ on a string, any boy, $b'$, on $g$'s string for any day $t' > t$ is at least as good as $b$.

**Proof:**

$P(k)$ - “boy on $g$’s string is at least as good as $b$ on day $t + k$”

$P(0)$– true. Girl has $b$ on string.

Assume $P(k)$. Let $b'$ be boy **on string** on day $t + k$. 
It gets better every day for girls..

**Improvement Lemma: It just gets better for girls.**
If on day $t$ a girl, $g$, has a boy $b$ on a string, any boy, $b'$, on $g$'s string for any day $t' > t$ is at least as good as $b$.

**Proof:**
$P(k)$ - “boy on $g$’s string is at least as good as $b$ on day $t + k$”

$P(0)$ – true. Girl has $b$ on string.

Assume $P(k)$. Let $b'$ be boy on string on day $t + k$.

On day $t + k + 1$, boy $b'$ comes back.
   Girl can choose $b'$,
Improvement Lemma: It just gets better for girls.
If on day $t$ a girl, $g$, has a boy $b$ on a string, any boy, $b'$, on $g$’s string for any day $t' > t$ is at least as good as $b$.

**Proof:**
$P(k)$ - “boy on $g$’s string is at least as good as $b$ on day $t + k$”

$P(0)$ – true. Girl has $b$ on string.

Assume $P(k)$. Let $b'$ be boy on string on day $t + k$.

On day $t + k + 1$, boy $b'$ comes back.
   Girl can choose $b'$, or do better with another boy, $b''$
**Improvement Lemma: It just gets better for girls.**

If on day $t$ a girl, $g$, has a boy $b$ on a string, any boy, $b'$, on $g$’s string for any day $t' > t$ is at least as good as $b$.

**Proof:**

$P(k)$ - “boy on $g$’s string is at least as good as $b$ on day $t + k$”

$P(0)$ – true. Girl has $b$ on string.

Assume $P(k)$. Let $b'$ be boy **on string** on day $t + k$.

On day $t + k + 1$, boy $b'$ comes back.

Girl can choose $b'$, or do better with another boy, $b''$

That is,
**Improvement Lemma:** It just gets better for girls.

If on day $t$ a girl, $g$, has a boy $b$ on a string, any boy, $b'$, on $g$'s string for any day $t' > t$ is at least as good as $b$.

**Proof:**

$P(k)$ - “boy on $g$’s string is at least as good as $b$ on day $t + k$”

$P(0)$– true. Girl has $b$ on string.

Assume $P(k)$. Let $b'$ be boy on string on day $t + k$.

On day $t + k + 1$, boy $b'$ comes back.

Girl can choose $b'$, or do better with another boy, $b''$

That is, $b \leq b'$ by induction hypothesis.
Improvement Lemma: It just gets better for girls.
If on day $t$ a girl, $g$, has a boy $b$ on a string, any boy, $b'$, on $g$’s string for any day $t' > t$ is at least as good as $b$.

Proof:
$P(k)$ - “boy on $g$’s string is at least as good as $b$ on day $t + k$”

$P(0)$ – true. Girl has $b$ on string.

Assume $P(k)$. Let $b'$ be boy on string on day $t + k$.

On day $t + k + 1$, boy $b'$ comes back.
   Girl can choose $b'$, or do better with another boy, $b''$

That is, $b \leq b'$ by induction hypothesis.
   And $b''$ is better than $b'$ by algorithm.
Improvement Lemma: It just gets better for girls.
If on day $t$ a girl, $g$, has a boy $b$ on a string, any boy, $b'$, on $g$'s string for any day $t' > t$ is at least as good as $b$.

Proof:
$P(k)$- - “boy on $g$’s string is at least as good as $b$ on day $t + k$”

$P(0)$– true. Girl has $b$ on string.

Assume $P(k)$. Let $b'$ be boy on string on day $t + k$.

On day $t + k + 1$, boy $b'$ comes back. 
  Girl can choose $b'$, or do better with another boy, $b''$

That is, $b \leq b'$ by induction hypothesis.
  And $b''$ is better than $b'$ by algorithm.

$P(k) \implies P(k + 1)$. 
**Improvement Lemma:** It just gets better for girls.

If on day \( t \) a girl, \( g \), has a boy \( b \) on a string, any boy, \( b' \), on \( g \)'s string for any day \( t' > t \) is at least as good as \( b \).

**Proof:**

\( P(k) \) - “boy on \( g \)'s string is at least as good as \( b \) on day \( t + k \)”

\( P(0) \) – true. Girl has \( b \) on string.

Assume \( P(k) \). Let \( b' \) be boy on string on day \( t + k \).

On day \( t + k + 1 \), boy \( b' \) comes back.

Girl can choose \( b' \), or do better with another boy, \( b'' \)

That is, \( b \leq b' \) by induction hypothesis.

And \( b'' \) is better than \( b' \) by algorithm.

\( P(k) \implies P(k + 1) \). And by principle of induction.
Improvement Lemma: It just gets better for girls.
If on day $t$ a girl, $g$, has a boy $b$ on a string, any boy, $b'$, on $g$’s string for any day $t' > t$ is at least as good as $b$.

Proof:
$P(k)$- “boy on $g$’s string is at least as good as $b$ on day $t + k$”

$P(0)$—true. Girl has $b$ on string.

Assume $P(k)$. Let $b'$ be boy on string on day $t + k$.

On day $t + k + 1$, boy $b'$ comes back.
- Girl can choose $b'$, or do better with another boy, $b''$

That is, $b \leq b'$ by induction hypothesis.
- And $b''$ is better than $b'$ by algorithm.

$P(k) \implies P(k + 1)$. And by principle of induction.
Lemma: Every boy is matched at end.
**Lemma:** Every boy is matched at end.

**Proof:**

---

**Pairing when done.**
Pairing when done.

Lemma: Every boy is matched at end.

Proof: If not, a boy $b$ must have been rejected $n$ times.
Lemma: Every boy is matched at end.

Proof:
If not, a boy $b$ must have been rejected $n$ times.

Every girl has been proposed to by $b$, 
**Lemma:** Every boy is matched at end.

**Proof:**
If not, a boy $b$ must have been rejected $n$ times.

Every girl has been proposed to by $b$, and *Improvement lemma*
Lemma: Every boy is matched at end.

Proof: 
If not, a boy $b$ must have been rejected $n$ times.

Every girl has been proposed to by $b$, and Improvement lemma

$\implies$ each girl has a boy on a string.
**Lemma:** Every boy is matched at end.

**Proof:**
If not, a boy $b$ must have been rejected $n$ times.

Every girl has been proposed to by $b$,
and **Improvement lemma**

$\implies$ each girl has a boy on a string.

and each boy on at most one string.
Lemma: Every boy is matched at end.

Proof: If not, a boy $b$ must have been rejected $n$ times.

Every girl has been proposed to by $b$, and Improvement lemma

$\implies$ each girl has a boy on a string.

and each boy on at most one string.

$n$ girls and $n$ boys.
Lemma: Every boy is matched at end.

Proof: If not, a boy $b$ must have been rejected $n$ times.

Every girl has been proposed to by $b$, and Improvement lemma

$\implies$ each girl has a boy on a string.

and each boy on at most one string.

$n$ girls and $n$ boys. Same number of each.
Lemma: Every boy is matched at end.

Proof:
If not, a boy $b$ must have been rejected $n$ times.

Every girl has been proposed to by $b$, and Improvement lemma

$\implies$ each girl has a boy on a string.

and each boy on at most one string.

$n$ girls and $n$ boys. Same number of each.
Lemma: Every boy is matched at end.

Proof:
If not, a boy $b$ must have been rejected $n$ times.

Every girl has been proposed to by $b$, and Improvement lemma

$\implies$ each girl has a boy on a string.

and each boy on at most one string.

$n$ girls and $n$ boys. Same number of each.

$\implies b$ must be on some girl’s string!
Lemma: Every boy is matched at end.

Proof:
If not, a boy $b$ must have been rejected $n$ times.

Every girl has been proposed to by $b$, and Improvement lemma

$\implies$ each girl has a boy on a string.

and each boy on at most one string.

$n$ girls and $n$ boys. Same number of each.

$\implies b$ must be on some girl’s string!

Contradiction.
Lemma: Every boy is matched at end.

Proof:
If not, a boy $b$ must have been rejected $n$ times.

Every girl has been proposed to by $b$, and Improvement lemma

$\implies$ each girl has a boy on a string.

and each boy on at most one string.

$n$ girls and $n$ boys. Same number of each.

$\implies b$ must be on some girl’s string!

Contradiction.
Lemma: There is no rogue couple for the pairing formed by traditional marriage algorithm.
**Lemma:** There is no rogue couple for the pairing formed by traditional marriage algorithm.

**Proof:**
Assume there is a rogue couple; \((b, g^*)\)
**Lemma:** There is no rogue couple for the pairing formed by traditional marriage algorithm.

**Proof:**
Assume there is a rogue couple; \((b, g^*)\)

\[
\begin{align*}
  b^* & \quad \longrightarrow \quad g^* \\
  b & \quad \longrightarrow \quad g
\end{align*}
\]
Lemma: There is no rogue couple for the pairing formed by traditional marriage algorithm.

Proof:  
Assume there is a rogue couple; \((b, g^*)\)

\[
\begin{align*}
& b^* \quad \text{-------------} \quad g^* \\
& b \quad \text{-------------} \quad g
\end{align*}
\]
Lemma: There is no rogue couple for the pairing formed by traditional marriage algorithm.

Proof: Assume there is a rogue couple; \((b, g^*)\)

\[ b^* \quad \longrightarrow \quad g^* \quad b \text{ likes } g^* \text{ more than } g. \]

\[ b \quad \longrightarrow \quad g \]
**Lemma:** There is no rogue couple for the pairing formed by traditional marriage algorithm.

**Proof:**
Assume there is a rogue couple; \((b, g^*)\)

\[
\begin{align*}
  b^* & \sim g^* \\
  b & \sim g
\end{align*}
\]

- \(b^*\) likes \(g^*\) more than \(g\).
- \(g^*\) likes \(b\) more than \(b^*\).
Lemma: There is no rogue couple for the pairing formed by traditional marriage algorithm.

Proof:
Assume there is a rogue couple; \((b, g^*)\)

\[
\begin{align*}
&b^* \quad \longrightarrow \quad g^* \\
&b \quad \longrightarrow \quad g
\end{align*}
\]

- \(b^*\) likes \(g^*\) more than \(g\).
- \(g^*\) likes \(b\) more than \(b^*\).

Boy \(b\) proposes to \(g^*\) before proposing to \(g\).
**Lemma:** There is no rogue couple for the pairing formed by traditional marriage algorithm.

**Proof:**
Assume there is a rogue couple; \((b, g^*)\)

\[
\begin{align*}
  b^* & \quad b \quad g^* \\
  b & \quad g \\
\end{align*}
\]

- \(b^*\) likes \(g^*\) more than \(g\).
- \(g^*\) likes \(b\) more than \(b^*\).

Boy \(b\) proposes to \(g^*\) before proposing to \(g\).

So \(g^*\) rejected \(b\) (since he moved on)
Pairing is Stable.

**Lemma:** There is no rogue couple for the pairing formed by traditional marriage algorithm.

**Proof:**
Assume there is a rogue couple; \((b, g^*)\)

\[
\begin{align*}
    & b^* \quad \longrightarrow \quad g^* \\
    & b \quad \longrightarrow \quad g
\end{align*}
\]

- \(b^*\) likes \(g^*\) more than \(g\).
- \(g^*\) likes \(b\) more than \(b^*\).

Boy \(b\) proposes to \(g^*\) before proposing to \(g\).

So \(g^*\) rejected \(b\) (since he moved on)

By improvement lemma, \(g^*\) likes \(b^*\) better than \(b\).
Lemma: There is no rogue couple for the pairing formed by traditional marriage algorithm.

Proof:
Assume there is a rogue couple; \((b, g^*)\)

\[\begin{align*}
  b^* & \sim g^* \quad b \text{ likes } g^* \text{ more than } g. \\
  b & \sim g \quad g^* \text{ likes } b \text{ more than } b^*.
\end{align*}\]

Boy \(b\) proposes to \(g^*\) before proposing to \(g\).

So \(g^*\) rejected \(b\) (since he moved on)

By improvement lemma, \(g^*\) likes \(b^*\) better than \(b\).

Contradiction!
**Lemma:** There is no rogue couple for the pairing formed by traditional marriage algorithm.

**Proof:**
Assume there is a rogue couple; \((b, g^*)\)

\[
\begin{align*}
  b^* & \bowtie g^* & b \text{ likes } g^* \text{ more than } g. \\
  b & \bowtie g & g^* \text{ likes } b \text{ more than } b^*.
\end{align*}
\]

Boy \(b\) proposes to \(g^*\) before proposing to \(g\).

So \(g^*\) rejected \(b\) (since he moved on)

By improvement lemma, \(g^*\) likes \(b^*\) better than \(b\).

*Contradiction!*
Good for boys? girls?

Is the TMA better for boys?
Good for boys? girls?

Is the TMA better for boys? for girls?
Is the TMA better for boys? for girls?

**Definition:** A pairing is $x$-optimal if $x$’s partner is its best partner in any stable pairing.
Is the TMA better for boys? for girls?

**Definition:** A *pairing is* \( x \)-**optimal** if \( x \)'s partner is its best partner in any *stable* pairing.

**Definition:** A *pairing is* \( x \)-**pessimal** if \( x \)'s partner is its worst partner in any *stable* pairing.
Is the TMA better for boys? for girls?

**Definition:** A pairing is \( x \)-**optimal** if \( x \)'s partner is its best partner in any **stable** pairing.

**Definition:** A pairing is \( x \)-**pessimal** if \( x \)'s partner is its worst partner in any **stable** pairing.

**Definition:** A pairing is boy **optimal** if it is \( x \)-optimal for all boys \( x \).
Good for boys? girls?

Is the TMA better for boys? for girls?

**Definition:** A pairing is $x$-optimal if $x$’s partner is its best partner in any stable pairing.

**Definition:** A pairing is $x$-pessimal if $x$’s partner is its worst partner in any stable pairing.

**Definition:** A pairing is boy optimal if it is $x$-optimal for all boys $x$.

..and so on for boy pessimal, girl optimal, girl pessimal.
Is the TMA better for boys? for girls?

**Definition:** A pairing is **x-optimal** if x’s partner is its best partner in any **stable** pairing.

**Definition:** A pairing is **x-pessimal** if x’s partner is its worst partner in any **stable** pairing.

**Definition:** A pairing is **boy optimal** if it is **x-optimal** for all boys x.

..and so on for boy pessimal, girl optimal, girl pessimal.

Claim: The optimal partner for a boy must be first in his preference list.
Is the TMA better for boys? for girls?

**Definition:** A pairing is *x-optimal* if *x*’s partner
is its best partner in any *stable* pairing.

**Definition:** A pairing is *x-pessimal* if *x*’s partner
is its worst partner in any *stable* pairing.

**Definition:** A pairing is **boy optimal** if it is *x-optimal* for all boys *x*.

..and so on for boy pessimal, girl optimal, girl pessimal.

Claim: The optimal partner for a boy must be first in his preference list.

True?
Is the TMA better for boys? for girls?

**Definition:** A pairing is **x-optimal** if $x'$'s partner is its best partner in any **stable** pairing.

**Definition:** A pairing is **x-pessimal** if $x'$'s partner is its worst partner in any **stable** pairing.

**Definition:** A pairing is **boy optimal** if it is $x$-optimal for all boys $x$.

..and so on for boy pessimal, girl optimal, girl pessimal.

Claim: The optimal partner for a boy must be first in his preference list.

True? False?
Good for boys? girls?

Is the TMA better for boys? for girls?

**Definition:** A pairing is *x-optimal* if *x*'s partner
is its best partner in any stable pairing.

**Definition:** A pairing is *x-pessimal* if *x*'s partner
is its worst partner in any stable pairing.

**Definition:** A pairing is **boy optimal** if it is *x-optimal* for all boys *x*.

..and so on for boy pessimal, girl optimal, girl pessimal.

Claim: The optimal partner for a boy must be first in his preference list.

True? False?  False!
Good for boys? girls?

Is the TMA better for boys? for girls?

**Definition:** A pairing is $x$-**optimal** if $x$’s partner is its best partner in any **stable** pairing.

**Definition:** A pairing is $x$-**pessimal** if $x$’s partner is its worst partner in any **stable** pairing.

**Definition:** A pairing is **boy optimal** if it is $x$-optimal for all boys $x$.

..and so on for boy pessimal, girl optimal, girl pessimal.

**Claim:** The optimal partner for a boy must be first in his preference list.

True? False? False!

Subtlety here: Best partner in any **stable** pairing.
Good for boys? girls?

Is the TMA better for boys? for girls?

**Definition:** A pairing is *x-optimal* if *x’s* partner is its best partner in any **stable** pairing.

**Definition:** A pairing is *x-pessimal* if *x’s* partner is its worst partner in any **stable** pairing.

**Definition:** A pairing is **boy optimal** if it is *x-optimal* for **all** boys *x*.

..and so on for boy pessimal, girl optimal, girl pessimal.

**Claim:** The optimal partner for a boy must be first in his preference list.

True? False? **False!**

Subtlety here: Best partner in any **stable** pairing. As well as you can in a globally stable solution!
Good for boys? girls?

Is the TMA better for boys? for girls?

**Definition:** A pairing is $x$-optimal if $x$’s partner is its best partner in any stable pairing.

**Definition:** A pairing is $x$-pessimal if $x$’s partner is its worst partner in any stable pairing.

**Definition:** A pairing is boy optimal if it is $x$-optimal for all boys $x$.

..and so on for boy pessimal, girl optimal, girl pessimal.

**Claim:** The optimal partner for a boy must be first in his preference list.

True? False? False!

Subtlety here: Best partner in any stable pairing. As well as you can in a globally stable solution!

**Question:** Is there a boy or girl optimal pairing?
Good for boys? girls?

Is the TMA better for boys? for girls?

**Definition:** A pairing is **x-optimal** if x’s partner is its best partner in any **stable** pairing.

**Definition:** A pairing is **x-pessimal** if x’s partner is its worst partner in any **stable** pairing.

**Definition:** A pairing is **boy optimal** if it is x-optimal for all boys x.

..and so on for boy pessimal, girl optimal, girl pessimal.

Claim: The optimal partner for a boy must be first in his preference list.

True? False? False!

Subtlety here: Best partner in any **stable** pairing.
As well as you can in a globally stable solution!

**Question:** Is there a boy or girl optimal pairing?
Is it possible:
Is the TMA better for boys? for girls?

**Definition:** A pairing is \( x \)-**optimal** if \( x \)'s partner is its best partner in any **stable** pairing.

**Definition:** A pairing is \( x \)-**pessimal** if \( x \)'s partner is its worst partner in any **stable** pairing.

**Definition:** A pairing is **boy optimal** if it is \( x \)-optimal for **all** boys \( x \).

..and so on for boy pessimal, girl optimal, girl pessimal.

**Claim:** The optimal partner for a boy must be first in his preference list.

True? False? False!

Subtlety here: Best partner in any **stable** pairing. As well as you can in a globally stable solution!

**Question:** Is there a boy or girl optimal pairing? Is it possible:

- \( b \)-optimal pairing different from the \( b' \)-optimal pairing!
Good for boys? girls?

Is the TMA better for boys? for girls?

**Definition:** A pairing is **x-optimal** if x’s partner is its best partner in any **stable** pairing.

**Definition:** A pairing is **x-pessimal** if x’s partner is its worst partner in any **stable** pairing.

**Definition:** A pairing is **boy optimal** if it is x-optimal for all boys x.

..and so on for boy pessimal, girl optimal, girl pessimal.

**Claim:** The optimal partner for a boy must be first in his preference list.

True? False? False!

Subtlety here: Best partner in any **stable** pairing. As well as you can in a globally stable solution!

**Question:** Is there a boy or girl optimal pairing?

Is it possible:

- b-optimal pairing different from the b'-optimal pairing!

Yes?
Is the TMA better for boys? for girls?

**Definition:** A pairing is **x-optimal** if x’s partner is its best partner in any stable pairing.

**Definition:** A pairing is **x-pessimal** if x’s partner is its worst partner in any stable pairing.

**Definition:** A pairing is **boy optimal** if it is x-optimal for all boys x.

..and so on for boy pessimal, girl optimal, girl pessimal.

**Claim:** The optimal partner for a boy must be first in his preference list.

   True? False?  False!

Subtlety here: Best partner in any stable pairing. As well as you can in a globally stable solution!

**Question:** Is there a boy or girl optimal pairing?

   Is it possible:
   
   b-optimal pairing different from the b’-optimal pairing!

Yes? No?
Is the TMA better for boys? for girls?

**Definition:** A pairing is *x-optimal* if *x*’s partner is its best partner in any stable pairing.

**Definition:** A pairing is *x-pessimal* if *x*’s partner is its worst partner in any stable pairing.

**Definition:** A pairing is boy optimal if it is *x-optimal* for all boys *x*.

..and so on for boy pessimal, girl optimal, girl pessimal.

**Claim:** The optimal partner for a boy must be first in his preference list.

True? False? False!

Subtlety here: Best partner in any stable pairing. As well as you can in a globally stable solution!

**Question:** Is there a boy or girl optimal pairing?

Is it possible:

- *b*-optimal pairing different from the *b’*-optimal pairing!

Yes? No?
TMA is optimal!

For boys?

Theorem: TMA produces a boy-optimal pairing.

Proof: Assume not: there are boys who do not get their optimal girl. Let $t$ be first day a boy $b$ gets rejected by his optimal girl $g$ who he is paired with in stable pairing $S$. $b^\ast$ knocks $b$ off of $g$'s string on day $t = \Rightarrow g$ prefers $b^\ast$ to $b$. By choice of $t$, $b^\ast$ prefers $g$ to optimal girl. $= \Rightarrow b^\ast$ prefers $g$ to his partner $g^\ast$ in $S$. Rogue couple for $S$. So $S$ is not a stable pairing. Contradiction.

Notes: $S$ - stable. $(b^\ast, g^\ast) \in S$. But $(b^\ast, g)$ is rogue couple! Used Well-Ordering principle... Induction.
TMA is optimal!

For boys? For girls?

Theorem: TMA produces a boy-optimal pairing.

Proof: Assume not: there are boys who do not get their optimal girl.

Let $t$ be first day a boy $b$ gets rejected by his optimal girl $g$ who he is paired with in stable pairing $S$.

$b^\ast$ knocks $b$ off of $g$'s string on day $t \Rightarrow g$ prefers $b^\ast$ to $b$.

By choice of $t$, $b^\ast$ prefers $g$ to optimal girl.

$\Rightarrow b^\ast$ prefers $g$ to his partner $g^\ast$ in $S$.

Rogue couple for $S$.

So $S$ is not a stable pairing.

Contradiction.

Notes: $S$ - stable.

$(b^\ast, g^\ast) \in S$.

But $(b^\ast, g^\ast)$ is rogue couple!

Used Well-Ordering principle... Induction.
For boys? For girls?

**Theorem:** TMA produces a boy-optimal pairing.
For boys? For girls?

**Theorem:** TMA produces a boy-optimal pairing.

**Proof:**

Assume not: there are boys who do not get their optimal girl.

Let \( t \) be first day a boy \( b \) gets rejected by his optimal girl \( g \) who he is paired with in stable pairing \( S \).

\( b^\ast \) knocks \( b \) off of \( g \)'s string on day \( t \) \( \Rightarrow \) \( g \) prefers \( b^\ast \) to \( b \).

By choice of \( t \), \( b^\ast \) prefers \( g \) to optimal girl \( \Rightarrow \) \( b^\ast \) prefers \( g \) to his partner \( g^\ast \) in \( S \).

Rogue couple for \( S \).

So \( S \) is not a stable pairing.

Contradiction.

**Notes:**

\( S \) - stable.

\((b^\ast, g^\ast) \in S \).

But \((b^\ast, g)\) is rogue couple!

Used Well-Ordering principle... Induction.
For boys? For girls?

**Theorem:** TMA produces a boy-optimal pairing.

**Proof:**
Assume not:

Notes:
$S$ - stable.
$((b^*, g^*)) \in S$.
But $(b^*, g^*)$ is a rogue couple!

Used Well-Ordering principle... Induction.
For boys? For girls?

**Theorem:** TMA produces a boy-optimal pairing.

**Proof:**
Assume not: there are boys who do not get their optimal girl.
For boys? For girls?

**Theorem:** TMA produces a boy-optimal pairing.

**Proof:**
Assume not: there are boys who do not get their optimal girl.

Let $t$ be first day a boy $b$ gets rejected
TMA is optimal!

For boys? For girls?

**Theorem:** TMA produces a boy-optimal pairing.

**Proof:**
Assume not: there are boys who do not get their optimal girl.

Let \( t \) be first day a boy \( b \) gets rejected
  by his optimal girl \( g \) who he is paired with

Notes:
- \( S \) - stable.
- \((b^\ast, g^\ast)\) \(\in S\).
- But \((b^\ast, g^\ast)\) is rogue couple!
- Used Well-Ordering principle...
- Induction.
For boys? For girls?

**Theorem:** TMA produces a boy-optimal pairing.

**Proof:**
Assume not: there are boys who do not get their optimal girl.

Let $t$ be first day a boy $b$ gets rejected
by his optimal girl $g$ who he is paired with
in stable pairing $S$.
For boys? For girls?

**Theorem:** TMA produces a boy-optimal pairing.

**Proof:**
Assume not: there are boys who do not get their optimal girl.

Let $t$ be first day a boy $b$ gets rejected
- by his optimal girl $g$ who he is paired with
- in stable pairing $S$.

$b^*$ - knocks $b$ off of $g$’s string on day $t$
For boys? For girls?

**Theorem:** TMA produces a boy-optimal pairing.

**Proof:**
Assume not: there are boys who do not get their optimal girl.

Let $t$ be first day a boy $b$ gets rejected
by his optimal girl $g$ who he is paired with
in stable pairing $S$.

$b^*$ - knocks $b$ off of $g$’s string on day $t \implies g$ prefers $b^*$ to $b$
For boys? For girls?

**Theorem:** TMA produces a boy-optimal pairing.

**Proof:**
Assume not: there are boys who do not get their optimal girl.

Let $t$ be first day a boy $b$ gets rejected
by his optimal girl $g$ who he is paired with
in stable pairing $S$.

$b^*$ - knocks $b$ off of $g$’s string on day $t \implies g$ prefers $b^*$ to $b$

By choice of $t$, $b^*$ prefers $g$ to optimal girl.
TMA is optimal!

For boys? For girls?

**Theorem:** TMA produces a boy-optimal pairing.

**Proof:**
Assume not: there are boys who do not get their optimal girl.

Let $t$ be first day a boy $b$ gets rejected
by his optimal girl $g$ who he is paired with
in stable pairing $S$.

$b^*$ - knocks $b$ off of $g$'s string on day $t \implies g$ prefers $b^*$ to $b$

By choice of $t$, $b^*$ prefers $g$ to optimal girl.

$\implies b^*$ prefers $g$ to his partner $g^*$ in $S$. 
For boys? For girls?

**Theorem:** TMA produces a boy-optimal pairing.

**Proof:**
Assume not: there are boys who do not get their optimal girl.

Let \( t \) be first day a boy \( b \) gets rejected
by his optimal girl \( g \) who he is paired with
in stable pairing \( S \).

\( b^* \) - knocks \( b \) off of \( g \)'s string on day \( t \) \( \implies \) \( g \) prefers \( b^* \) to \( b \)

By choice of \( t \), \( b^* \) prefers \( g \) to optimal girl.

\( \implies b^* \) prefers \( g \) to his partner \( g^* \) in \( S \).

Rogue couple for \( S \).
TMA is optimal!

For boys? For girls?

**Theorem:** TMA produces a boy-optimal pairing.

**Proof:**
Assume not: there are boys who do not get their optimal girl.

Let $t$ be first day a boy $b$ gets rejected
by his optimal girl $g$ who he is paired with
in stable pairing $S$.

$b^*$ - knocks $b$ off of $g$’s string on day $t \implies g$ prefers $b^*$ to $b$

By choice of $t$, $b^*$ prefers $g$ to optimal girl.

$\implies b^*$ prefers $g$ to his partner $g^*$ in $S$.

Rogue couple for $S$.  
So $S$ is not a stable pairing.
For boys? For girls?

**Theorem:** TMA produces a boy-optimal pairing.

**Proof:**
Assume not: there are boys who do not get their optimal girl.

Let $t$ be first day a boy $b$ gets rejected
by his optimal girl $g$ who he is paired with
in stable pairing $S$.

$b^*$ - knocks $b$ off of $g$’s string on day $t \implies g$ prefers $b^*$ to $b$

By choice of $t$, $b^*$ prefers $g$ to optimal girl.

$\implies b^*$ prefers $g$ to his partner $g^*$ in $S$.

Rogue couple for $S$.
So $S$ is not a stable pairing. Contradiction.
TMA is optimal!

For boys? For girls?

**Theorem:** TMA produces a boy-optimal pairing.

**Proof:**
Assume not: there are boys who do not get their optimal girl.

Let $t$ be first day a boy $b$ gets rejected by his optimal girl $g$ who he is paired with in stable pairing $S$.

$b^*$ - knocks $b$ off of $g$’s string on day $t \implies g$ prefers $b^*$ to $b$

By choice of $t$, $b^*$ prefers $g$ to optimal girl.

$\implies b^*$ prefers $g$ to his partner $g^*$ in $S$.

Rogue couple for $S$.
So $S$ is not a stable pairing. Contradiction.
**TMA is optimal!**

For boys? For girls?

**Theorem:** TMA produces a boy-optimal pairing.

**Proof:**
Assume not: there are boys who do not get their optimal girl.

Let $t$ be first day a boy $b$ gets rejected
by his optimal girl $g$ who he is paired with
in stable pairing $S$.

$b^*$ - knocks $b$ off of $g$’s string on day $t \implies g$ prefers $b^*$ to $b$

By choice of $t$, $b^*$ prefers $g$ to optimal girl.

$\implies b^*$ prefers $g$ to his partner $g^*$ in $S$.

Rogue couple for $S$.
So $S$ is not a stable pairing. Contradiction.

Notes:
For boys? For girls?

**Theorem:** TMA produces a boy-optimal pairing.

**Proof:**
Assume not: there are boys who do not get their optimal girl.

Let \( t \) be first day a boy \( b \) gets rejected
by his optimal girl \( g \) who he is paired with
in stable pairing \( S \).

\(b^*\) - knocks \( b\) off of \( g\)'s string on day \( t \) \(\implies\) \( g \) prefers \( b^* \) to \( b \)

By choice of \( t \), \( b^* \) prefers \( g \) to optimal girl.

\(\implies\) \( b^* \) prefers \( g \) to his partner \( g^* \) in \( S \).

Rogue couple for \( S \).
So \( S \) is not a stable pairing. Contradiction. 

Notes: \( S \) - stable.
For boys? For girls?

**Theorem:** TMA produces a boy-optimal pairing.

**Proof:**
Assume not: there are boys who do not get their optimal girl.

Let \( t \) be first day a boy \( b \) gets rejected
by his optimal girl \( g \) who he is paired with
in stable pairing \( S \).

\( b^* \) - knocks \( b \) off of \( g \)'s string on day \( t \) \( \implies \) \( g \) prefers \( b^* \) to \( b \)

By choice of \( t \), \( b^* \) prefers \( g \) to optimal girl.

\( \implies b^* \) prefers \( g \) to his partner \( g^* \) in \( S \).

**Rogue couple for \( S \).**
**So \( S \) is not a stable pairing. Contradiction.**

Notes: \( S \) - stable. \((b^*, g^*)\) \( \in \) \( S \).
TMA is optimal!

For boys? For girls?

**Theorem:** TMA produces a boy-optimal pairing.

**Proof:**
Assume not: there are boys who do not get their optimal girl.

Let $t$ be first day a boy $b$ gets rejected
by his optimal girl $g$ who he is paired with
in stable pairing $S$.

$b^*$ - knocks $b$ off of $g$'s string on day $t \implies g$ prefers $b^*$ to $b$

By choice of $t$, $b^*$ prefers $g$ to optimal girl.

$\implies b^*$ prefers $g$ to his partner $g^*$ in $S$.

**Rogue couple for $S$.**
So $S$ is not a stable pairing. **Contradiction.**

Notes: $S$ - stable. $(b^*, g^*) \in S$. But $(b^*, g)$
TMA is optimal!

For boys? For girls?

**Theorem:** TMA produces a boy-optimal pairing.

**Proof:**
Assume not: there are boys who do not get their optimal girl.

Let \( t \) be first day a boy \( b \) gets rejected
by his optimal girl \( g \) who he is paired with
in stable pairing \( S \).

\( b^* \) - knocks \( b \) off of \( g \)'s string on day \( t \) \( \implies \) \( g \) prefers \( b^* \) to \( b \)

By choice of \( t \), \( b^* \) prefers \( g \) to optimal girl.

\( \implies b^* \) prefers \( g \) to his partner \( g^* \) in \( S \).

Rogue couple for \( S \).
So \( S \) is not a stable pairing. Contradiction.

Notes: \( S \) - stable. \( (b^*, g^*) \in S \). But \( (b^*, g) \) is rogue couple!
For boys? For girls?

**Theorem:** TMA produces a boy-optimal pairing.

**Proof:**
Assume not: there are boys who do not get their optimal girl.

Let \( t \) be first day a boy \( b \) gets rejected by his optimal girl \( g \) who he is paired with in stable pairing \( S \).

\( b^* \) - knocks \( b \) off of \( g \)'s string on day \( t \) \( \implies \) \( g \) prefers \( b^* \) to \( b \)

By choice of \( t \), \( b^* \) prefers \( g \) to optimal girl.

\( \implies \) \( b^* \) prefers \( g \) to his partner \( g^* \) in \( S \).

Rogue couple for \( S \).
So \( S \) is not a stable pairing. Contradiction.

Notes: \( S \) - stable. \((b^*,g^*) \in S \). But \((b^*,g)\) is rogue couple!

Used Well-Ordering principle...
For boys? For girls?

**Theorem:** TMA produces a boy-optimal pairing.

**Proof:**
Assume not: there are boys who do not get their optimal girl.

Let $t$ be first day a boy $b$ gets rejected by his optimal girl $g$ who he is paired with in stable pairing $S$.

$b^* -$ knocks $b$ off of $g$’s string on day $t \implies g$ prefers $b^*$ to $b$

By choice of $t$, $b^*$ prefers $g$ to optimal girl.

$\implies b^*$ prefers $g$ to his partner $g^*$ in $S$.

**Rogue couple for $S$.**
**So $S$ is not a stable pairing. Contradiction.**

Notes: $S$ - stable. $(b^*, g^*) \in S$. But $(b^*, g)$ is rogue couple!

Used Well-Ordering principle...Induction.
How about for girls?

Theorem:

TMA produces girl-pessimal pairing.

T – pairing produced by TMA.
S – worse stable pairing for girl g.

In T, (g, b) is pair.
In S, (g, b∗) is pair.

g likes b∗ less than she likes b.

T is boy optimal, so b likes g more than his partner in S.

S is not stable.

Contradiction.

Notes:
Not really induction.

Structural statement: Boy optimality =⇒ Girl pessimality.
How about for girls?

**Theorem:** TMA produces girl-pessimal pairing.
How about for girls?

Theorem: TMA produces girl-pessimal pairing.

$T$ – pairing produced by TMA.
How about for girls?

**Theorem:** TMA produces girl-pessimal pairing.

$T$ – pairing produced by TMA.

$S$ – worse stable pairing for girl $g$. 
Theorem: TMA produces girl-pessimal pairing.

$T$ – pairing produced by TMA.

$S$ – worse stable pairing for girl $g$.

In $T$, $(g, b)$ is pair.
Theorem: TMA produces girl-pessimal pairing.

$T$ – pairing produced by TMA.

$S$ – worse stable pairing for girl $g$.

In $T$, $(g, b)$ is pair.

In $S$, $(g, b^*)$ is pair.
How about for girls?

**Theorem:** TMA produces girl-pessimal pairing.

*T* – pairing produced by TMA.

*S* – worse stable pairing for girl *g*.

In *T*, *(g, b)* is pair.

In *S*, *(g, b*) is pair.

*g* likes *b* less than she likes *b*.
How about for girls?

**Theorem:** TMA produces girl-pessimal pairing.

$T$ – pairing produced by TMA.

$S$ – worse stable pairing for girl $g$.

In $T$, $(g, b)$ is pair.

In $S$, $(g, b^*)$ is pair.

$g$ likes $b^*$ less than she likes $b$.

$T$ is boy optimal, so $b$ likes $g$ more than his partner in $S$. 
How about for girls?

**Theorem:** TMA produces girl-pessimal pairing.

$T$ – pairing produced by TMA.

$S$ – worse stable pairing for girl $g$.

In $T$, $(g,b)$ is pair.

In $S$, $(g,b^*)$ is pair.

$g$ likes $b^*$ less than she likes $b$.

$T$ is boy optimal, so $b$ likes $g$ more than his partner in $S$.

$(g,b)$ is Rogue couple for $S$
How about for girls?

**Theorem:** TMA produces girl-pessimal pairing.

$T$ – pairing produced by TMA.

$S$ – worse stable pairing for girl $g$.

In $T$, $(g, b)$ is pair.

In $S$, $(g, b^*)$ is pair.

$g$ likes $b^*$ less than she likes $b$.

$T$ is boy optimal, so $b$ likes $g$ more than his partner in $S$.

$(g, b)$ is Rogue couple for $S$

$S$ is not stable.
How about for girls?

**Theorem:** TMA produces girl-pessimal pairing.

* T – pairing produced by TMA.

* S – worse **stable pairing** for girl *g*.

In *T*, *(g, b)* is pair.

In *S*, *(g, b*) is pair.

*g* likes *b* less than she likes *b*.

*T* is boy optimal, so *b* likes *g* more than his partner in *S*.

*(g, b)* is Rogue couple for *S*

* S is not stable.

Contradiction.
How about for girls?

**Theorem:** TMA produces girl-pessimal pairing.

$T$ – pairing produced by TMA.

$S$ – worse stable pairing for girl $g$.

In $T$, $(g, b)$ is pair.

In $S$, $(g, b^*)$ is pair.

$g$ likes $b^*$ less than she likes $b$.

$T$ is boy optimal, so $b$ likes $g$ more than his partner in $S$.

$(g, b)$ is Rogue couple for $S$

$S$ is not stable.

**Contradiction.**
How about for girls?

**Theorem:** TMA produces girl-pessimal pairing.

$T$ – pairing produced by TMA.

$S$ – worse stable pairing for girl $g$.

In $T$, $(g,b)$ is pair.

In $S$, $(g,b^*)$ is pair.

$g$ likes $b^*$ less than she likes $b$.

$T$ is boy optimal, so $b$ likes $g$ more than his partner in $S$.

$(g,b)$ is Rogue couple for $S$

$S$ is not stable.

**Contradiction.**

Notes:
Theorem: TMA produces girl-pessimal pairing.

$T$ – pairing produced by TMA.

$S$ – worse stable pairing for girl $g$.

In $T$, $(g, b)$ is pair.

In $S$, $(g, b^*)$ is pair.

$g$ likes $b^*$ less than she likes $b$.

$T$ is boy optimal, so $b$ likes $g$ more than his partner in $S$.

$(g, b)$ is Rogue couple for $S$.

$S$ is not stable.

Contradiction.

Notes: Not really induction.
How about for girls?

**Theorem:** TMA produces girl-pessimal pairing.

$T$ – pairing produced by TMA.

$S$ – worse stable pairing for girl $g$.

In $T$, $(g, b)$ is pair.

In $S$, $(g, b^*)$ is pair.

$g$ likes $b^*$ less than she likes $b$.

$T$ is boy optimal, so $b$ likes $g$ more than his partner in $S$.

$(g, b)$ is Rogue couple for $S$

$S$ is not stable.

Contradiction.

Notes: Not really induction.

   Structural statement: Boy optimality
How about for girls?

**Theorem:** TMA produces girl-pessimal pairing.

$T$ – pairing produced by TMA.

$S$ – worse stable pairing for girl $g$.

In $T$, $(g, b)$ is pair.

In $S$, $(g, b^*)$ is pair.

$g$ likes $b^*$ less than she likes $b$.

$T$ is boy optimal, so $b$ likes $g$ more than his partner in $S$.

$(g, b)$ is Rogue couple for $S$

$S$ is not stable.

**Contradiction.**

Notes: Not really induction.

Structural statement: Boy optimality $\implies$ Girl pessimality.
Quick Questions.

How does one make it better for girls?
How does one make it better for girls?

SMA - stable marriage algorithm. One side proposes.
Quick Questions.

How does one make it better for girls?

SMA - stable marriage algorithm. One side proposes.
TMA - boys propose.
Quick Questions.

How does one make it better for girls?

SMA - stable marriage algorithm. One side proposes.
TMA - boys propose.
Girls could propose.
Quick Questions.

How does one make it better for girls?

SMA - stable marriage algorithm. One side proposes.
TMA - boys propose.
Girls could propose. $\implies$ optimal for girls.
Residency Matching..

The method was used to match residents to hospitals. Hospital optimal until 1990's. Resident optimal.
The method was used to match residents to hospitals.
The method was used to match residents to hospitals.

Hospital optimal....
The method was used to match residents to hospitals.

Hospital optimal....

..until 1990’s...
The method was used to match residents to hospitals.

Hospital optimal....

..until 1990’s...Resident optimal.
Summary

Tomorrow Alex starts on Infinity and Countability

Thank you all!
Tomorrow Alex starts on Infinity and Countability
Tomorrow Alex starts on Infinity and Countability

Thank you all!