CS70: Discrete Math and Probability

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June 27, 2016
More graphs
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Connectivity
Eulerian Tour
Planar graphs
  5 coloring theorem
$u$ and $v$ are connected if there is a path between $u$ and $v$. 

A connected graph is a graph where all pairs of vertices are connected.
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If one vertex $x$ is connected to every other vertex.
Connectivity

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Is graph connected?
Connectivity

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Is graph connected? Yes?
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Is graph connected? Yes? No?
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Proof idea:
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A connected graph is a graph where all pairs of vertices are connected.

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   Is graph connected? Yes? No?

Proof idea: Use path from $u$ to $x$ and then from $x$ to $v$. 


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May not be simple!
$u$ and $v$ are connected if there is a path between $u$ and $v$.

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Is graph connected? Yes? No?

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May not be simple!
Either modify definition to walk.
$u$ and $v$ are **connected** if there is a path between $u$ and $v$.

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Either modify definition to walk.
Or cut out cycles.
Is graph above connected?

Connected Components:

- \{1, 10, 7, 5, 8, 4, 3, 11\}
- \{2, 9, 6\}

Connected component - maximal set of connected vertices.

Quick Check: Is \{10, 7, 5\} a connected component?

No.
Is graph above connected? Yes!

Connected Components:
- \{1\}, \{10, 7, 5\}, \{11, 4, 3, 8\}, \{2, 6, 9\}.
Connected component

Is graph above connected? Yes!

How about now?
Is graph above connected? Yes!

How about now? No!
Is graph above connected? Yes!

How about now? No!

Connected Components?
Is graph above connected? Yes!

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**Connected Components?** \{1\}, \{10, 7, 5, 8, 4, 3, 11\}, \{2, 9, 6\}.
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Is graph above connected? Yes!

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**Connected Components**? \{1\}, \{10, 7, 5, 8, 4, 3, 11\}, \{2, 9, 6\}.

Connected component - maximal set of connected vertices.

Quick Check: Is \{10, 7, 5\} a connected component? No.
Finally..back to bridges!

**Definition:**
An Eulerian Tour is a tour that visits each edge exactly once.

**Theorem:**
Any undirected graph has an Eulerian tour if and only if all vertices have even degree and is connected.

**Proof of only if:**
Eulerian $\implies$ connected and all even degree.

Eulerian Tour is connected so graph is connected.

Tour enters and leaves vertex $v$ on each visit.

Uses two incident edges per visit.

Therefore $v$ has even degree.

When you enter, you leave.

For starting node, tour leaves first...then enters at end.
Definition:

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When you enter, you leave. For starting node, tour leaves first ....then enters at end.
Proof of if: Even + connected $\implies$ Eulerian Tour.
We will give an algorithm.
Finding a tour!

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... till you get back to $v$.
Proof of if: Even + connected $\Rightarrow$ Eulerian Tour.
We will give an algorithm. First by picture.

1. Take a walk starting from $v$ (1) on “unused” edges 
... till you get back to $v$.
2. Remove tour, $C$.
Proof of if: Even + connected $\implies$ Eulerian Tour.
We will give an algorithm. First by picture.

1. Take a walk starting from $v$ (1) on “unused” edges ... till you get back to $v$.
2. Remove tour, $C$.
3. Let $G_1, \ldots, G_k$ be connected components.

1, 10 7, 8 4 11, 5 3 9 6 2, 11, 4 5, 2, 10
Proof of if: Even + connected $\implies$ Eulerian Tour.

We will give an algorithm. First by picture.

1. Take a walk starting from $v(1)$ on “unused” edges ... till you get back to $v$.
2. Remove tour, $C$.
3. Let $G_1, \ldots, G_k$ be connected components.
   Each is touched by $C$. 
4. Recurse on $G_1, \ldots, G_k$ starting from $v_i$.
5. Splice together.
Finding a tour!

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   ... till you get back to $v$.
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3. Let $G_1, \ldots, G_k$ be connected components.
   Each is touched by $C$.
   Why?

1 2 3 4 5 6 7 8 9 10 11
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   Each is touched by $C$.
   Why? $G$ was connected.
   Let $v_i$ be (first) node in $G_i$ touched by $C$. 

[Diagram of a graph with nodes and edges indicating the Eulerian tour and the steps of the algorithm.]
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   Example: $v_1 = 1$, 

![Graph Image]
Proof of if: Even + connected $\implies$ Eulerian Tour.

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2. Remove tour, $C$.
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   Each is touched by $C$.
   Why? $G$ was connected.
   Let $v_i$ be (first) node in $G_i$ touched by $C$.
   Example: $v_1 = 1$, $v_2 = 10$, 

![Diagram of a graph with labeled nodes and arrows indicating the Eulerian Tour path.](image-url)
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   Example: $v_1 = 1$, $v_2 = 10$, $v_3 = 4$, $v_4 = 2$. 

![Diagram of a graph with nodes and edges labeled 1 to 11]
Proof of if: Even + connected $\implies$ Eulerian Tour.
We will give an algorithm. First by picture.

1. Take a walk starting from $v$ (1) on “unused” edges ... till you get back to $v$.
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   Why? $G$ was connected.
   Let $v_i$ be (first) node in $G_i$ touched by $C$.
   Example: $v_1 = 1, v_2 = 10, v_3 = 4, v_4 = 2$.
4. Recurse on $G_1, \ldots, G_k$ starting from $v_i$
Proof of if: Even + connected $\implies$ Eulerian Tour.
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1. Take a walk starting from $v$ (1) on “unused” edges
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   Example: $v_1 = 1$, $v_2 = 10$, $v_3 = 4$, $v_4 = 2$.
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   1, 10
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   $1, 10, 7, 8, 5, 10, 8, 4, 3, 11, 4, 5, 2, 6, 9, 2$
Finding a tour!

**Proof of if: Even + connected $\implies$ Eulerian Tour.**

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5. Splice together.
   $1, 10, 7, 8, 5, 10, 8, 4, 3, 11, 4, 5, 2, 6, 9, 2$ and to 1!
1. Take a walk from arbitrary node $v$, until you get back to $v$. 
Finding a tour: in general.

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**Claim:** Do get back to $v$!
Finding a tour: in general.

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**Claim:** Do get back to $v$!

**Proof of Claim:** Even degree.
Finding a tour: in general.

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**Claim:** Do get back to $v$!

**Proof of Claim:** Even degree. If enter, can leave
Finding a tour: in general.

1. Take a walk from arbitrary node \( v \), until you get back to \( v \).

**Claim**: Do get back to \( v \)!
**Proof of Claim**: Even degree. If enter, can leave except for \( v \).

2. Remove cycle, \( C \), from \( G \).
   Resulting graph may be disconnected. (Removed edges!)
3. Find tour \( T_i \) of \( G_i \) starting/ending at \( v_i \).
4. Splice \( T_i \) into \( C \) where \( v_i \) first appears in \( C \).

Visits every edge once:
Visits edges in \( C \) exactly once.
By induction for all edges in each \( G_i \).
Finding a tour: in general.

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2. Remove cycle, \( C \), from \( G \).
Resulting graph may be disconnected. (Removed edges!)
Let components be \( G_1, \ldots, G_k \).
1. Take a walk from arbitrary node \( v \), until you get back to \( v \).

**Claim:** Do get back to \( v \)!

**Proof of Claim:** Even degree. If enter, can leave except for \( v \).

2. Remove cycle, \( C \), from \( G \).
   - Resulting graph may be disconnected. (Removed edges!)
   - Let components be \( G_1, \ldots, G_k \).
   - Let \( v_i \) be first vertex of \( C \) that is in \( G_i \).
Finding a tour: in general.

1. Take a walk from arbitrary node $v$, until you get back to $v$.

**Claim:** Do get back to $v$!

**Proof of Claim:** Even degree. If enter, can leave except for $v$.

2. Remove cycle, $C$, from $G$.

Resulting graph may be disconnected. (Removed edges!)

Let components be $G_1, \ldots, G_k$.

Let $v_i$ be first vertex of $C$ that is in $G_i$.

Why is there a $v_i$ in $C$?
Finding a tour: in general.

1. Take a walk from arbitrary node \( v \), until you get back to \( v \).

**Claim:** Do get back to \( v \)!

**Proof of Claim:** Even degree. If enter, can leave except for \( v \). \( \square \)

2. Remove cycle, \( C \), from \( G \).
   Resulting graph may be disconnected. (Removed edges!)
   Let components be \( G_1, \ldots, G_k \).
   Let \( v_i \) be first vertex of \( C \) that is in \( G_i \).
   Why is there a \( v_i \) in \( C \)?
   \( G \) was connected \( \implies \)
Finding a tour: in general.

1. Take a walk from arbitrary node $v$, until you get back to $v$.

**Claim:** Do get back to $v$!

**Proof of Claim:** Even degree. If enter, can leave except for $v$. □

2. Remove cycle, $C$, from $G$.

Resulting graph may be disconnected. (Removed edges!)

Let components be $G_1, \ldots, G_k$.

Let $v_i$ be first vertex of $C$ that is in $G_i$.

Why is there a $v_i$ in $C$?

$G$ was connected $\implies$

a vertex in $G_i$ must be incident to a removed edge in $C$. 
Finding a tour: in general.

1. Take a walk from arbitrary node \( v \), until you get back to \( v \).

**Claim:** Do get back to \( v \)!

**Proof of Claim:** Even degree. If enter, can leave except for \( v \).

2. Remove cycle, \( C \), from \( G \).

Resulting graph may be disconnected. (Removed edges!)

Let components be \( G_1, \ldots, G_k \).

Let \( v_i \) be first vertex of \( C \) that is in \( G_i \).

Why is there a \( v_i \) in \( C \)?

- \( G \) was connected \( \implies \)
  - a vertex in \( G_i \) must be incident to a removed edge in \( C \).
Finding a tour: in general.

1. Take a walk from arbitrary node \(v\), until you get back to \(v\).

   **Claim:** Do get back to \(v\)!

   **Proof of Claim:** Even degree. If enter, can leave except for \(v\).

2. Remove cycle, \(C\), from \(G\).

   Resulting graph may be disconnected. (Removed edges!)

   Let components be \(G_1, \ldots, G_k\).

   Let \(v_i\) be first vertex of \(C\) that is in \(G_i\).

   **Why is there a \(v_i\) in \(C\)?**

   - \(G\) was connected \(\implies\)
     - a vertex in \(G_i\) must be incident to a removed edge in \(C\).

   **Claim:** Each vertex in each \(G_i\) has even degree
Finding a tour: in general.

1. Take a walk from arbitrary node \( v \), until you get back to \( v \).

Claim: Do get back to \( v \)!

Proof of Claim: Even degree. If enter, can leave except for \( v \).  

2. Remove cycle, \( C \), from \( G \).

Resulting graph may be disconnected. (Removed edges!)

Let components be \( G_1, \ldots, G_k \).

Let \( v_i \) be first vertex of \( C \) that is in \( G_i \).

Why is there a \( v_i \) in \( C \)?

\( G \) was connected \( \implies \) a vertex in \( G_i \) must be incident to a removed edge in \( C \).

Claim: Each vertex in each \( G_i \) has even degree and is connected.
Finding a tour: in general.

1. Take a walk from arbitrary node \( v \), until you get back to \( v \).

**Claim:** Do get back to \( v \)!

**Proof of Claim:** Even degree. If enter, can leave except for \( v \).

2. Remove cycle, \( C \), from \( G \).
   Resulting graph may be disconnected. (Removed edges!)
   Let components be \( G_1, \ldots, G_k \).
   Let \( v_i \) be first vertex of \( C \) that is in \( G_i \).
   Why is there a \( v_i \) in \( C \)?
   \( G \) was connected \( \implies \) a vertex in \( G_i \) must be incident to a removed edge in \( C \).

**Claim:** Each vertex in each \( G_i \) has even degree and is connected.

**Prf:** Tour \( C \) has even incidences to any vertex \( v \).
Finding a tour: in general.

1. Take a walk from arbitrary node \( v \), until you get back to \( v \).

**Claim:** Do get back to \( v \)!
**Proof of Claim:** Even degree. If enter, can leave except for \( v \).

2. Remove cycle, \( C \), from \( G \).
Resulting graph may be disconnected. (Removed edges!)
Let components be \( G_1, \ldots, G_k \).
Let \( v_i \) be first vertex of \( C \) that is in \( G_i \).
Why is there a \( v_i \) in \( C \)?
   \( G \) was connected \( \implies \)
   a vertex in \( G_i \) must be incident to a removed edge in \( C \).

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Finding a tour: in general.

1. Take a walk from arbitrary node $v$, until you get back to $v$.

**Claim:** Do get back to $v$!

**Proof of Claim:** Even degree. If enter, can leave except for $v$. □

2. Remove cycle, $C$, from $G$.

Resulting graph may be disconnected. (Removed edges!)

Let components be $G_1, \ldots, G_k$.

Let $v_i$ be first vertex of $C$ that is in $G_i$.

Why is there a $v_i$ in $C$?

$G$ was connected $\implies$

a vertex in $G_i$ must be incident to a removed edge in $C$.

**Claim:** Each vertex in each $G_i$ has even degree and is connected.

**Prf:** Tour $C$ has even incidences to any vertex $v$. □

3. Find tour $T_i$ of $G_i$
Finding a tour: in general.

1. Take a walk from arbitrary node \( v \), until you get back to \( v \).

**Claim:** Do get back to \( v \)!

**Proof of Claim:** Even degree. If enter, can leave except for \( v \).

2. Remove cycle, \( C \), from \( G \).
   Resulting graph may be disconnected. (Removed edges!)
   Let components be \( G_1, \ldots, G_k \).
   Let \( v_i \) be first vertex of \( C \) that is in \( G_i \).
   Why is there a \( v_i \) in \( C \)?
   \[
   G \text{ was connected} \implies \text{a vertex in } G_i \text{ must be incident to a removed edge in } C.
   \]

**Claim:** Each vertex in each \( G_i \) has even degree and is connected.

**Prf:** Tour \( C \) has even incidences to any vertex \( v \).

3. Find tour \( T_i \) of \( G_i \) starting/ending at \( v_i \).
Finding a tour: in general.

1. Take a walk from arbitrary node $v$, until you get back to $v$.

**Claim:** Do get back to $v$!

**Proof of Claim:** Even degree. If enter, can leave except for $v$. □

2. Remove cycle, $C$, from $G$.

Resulting graph may be disconnected. (Removed edges!)

Let components be $G_1, \ldots, G_k$.

Let $v_i$ be first vertex of $C$ that is in $G_i$.

Why is there a $v_i$ in $C$?

- $G$ was connected $\implies$
  - a vertex in $G_i$ must be incident to a removed edge in $C$.

**Claim:** Each vertex in each $G_i$ has even degree and is connected.

**Prf:** Tour $C$ has even incidences to any vertex $v$. □

3. Find tour $T_i$ of $G_i$ starting/ending at $v_i$. Induction.
Finding a tour: in general.

1. Take a walk from arbitrary node \( v \), until you get back to \( v \).

**Claim:** Do get back to \( v \)!

**Proof of Claim:** Even degree. If enter, can leave except for \( v \).

2. Remove cycle, \( C \), from \( G \).

Resulting graph may be disconnected. (Removed edges!)

Let components be \( G_1, \ldots, G_k \).

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3. Find tour \( T_i \) of \( G_i \) starting/ending at \( v_i \). Induction.

4. Splice \( T_i \) into \( C \) where \( v_i \) first appears in \( C \).
Finding a tour: in general.

1. Take a walk from arbitrary node \( v \), until you get back to \( v \).

**Claim:** Do get back to \( v \)!
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Resulting graph may be disconnected. (Removed edges!)
Let components be \( G_1, \ldots, G_k \).
Let \( v_i \) be first vertex of \( C \) that is in \( G_i \).
   Why is there a \( v_i \) in \( C \)?
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**Prf:** Tour \( C \) has even incidences to any vertex \( v \).

3. Find tour \( T_i \) of \( G_i \) starting/ending at \( v_i \). Induction.
4. Splice \( T_i \) into \( C \) where \( v_i \) first appears in \( C \).

Visits every edge once:
   Visits edges in \( C \)
Finding a tour: in general.

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Visits every edge once:

Visits edges in \( C \) exactly once.
Finding a tour: in general.

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Visits every edge once:

- Visits edges in \( C \) exactly once.
- By induction for all edges in each \( G_i \).
Finding a tour: in general.

1. Take a walk from arbitrary node $v$, until you get back to $v$.

**Claim:** Do get back to $v$!

**Proof of Claim:** Even degree. If enter, can leave except for $v$.

2. Remove cycle, $C$, from $G$.

Resulting graph may be disconnected. (Removed edges!)

Let components be $G_1, \ldots, G_k$.

Let $v_i$ be first vertex of $C$ that is in $G_i$.

Why is there a $v_i$ in $C$?

$G$ was connected $\implies$ a vertex in $G_i$ must be incident to a removed edge in $C$.

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4. Splice $T_i$ into $C$ where $v_i$ first appears in $C$.

Visits every edge once:

- Visits edges in $C$ exactly once.
- By induction for all edges in each $G_i$.
Planar graphs.

A graph that can be drawn in the plane without edge crossings.
Planar graphs.

A graph that can be drawn in the plane without edge crossings.
Planar graphs.

A graph that can be drawn in the plane without edge crossings.

Planar? Yes for Triangle.
Planar graphs.

A graph that can be drawn in the plane without edge crossings.

Planar? Yes for Triangle.
Four node complete?

Planar? Yes for Triangle.
Four node complete?
Planar graphs.

A graph that can be drawn in the plane without edge crossings.

Planar? Yes for Triangle.
Four node complete? Yes.
Planar graphs.

A graph that can be drawn in the plane without edge crossings.

Planar? Yes for Triangle.
Four node complete? Yes.
Five node complete or $K_5$? No!
Planar graphs.

A graph that can be drawn in the plane without edge crossings.

Planar? Yes for Triangle.
Four node complete? Yes.
Five node complete or $K_5$? No!
Planar graphs.

A graph that can be drawn in the plane without edge crossings.

Planar?  Yes for Triangle.
Four node complete?  Yes.
Five node complete or $K_5$?  No! Why?
Planar graphs.

A graph that can be drawn in the plane without edge crossings.

Planar? Yes for Triangle.
Four node complete? Yes.
Five node complete or $K_5$? No! Why? Later.
Planar graphs.

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Four node complete? Yes.
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Planar graphs.

A graph that can be drawn in the plane without edge crossings.

Planar? Yes for Triangle.
Four node complete? Yes.
Five node complete or $K_5$? No! Why? Later.

Two to three nodes, bipartite?
A graph that can be drawn in the plane without edge crossings.

Planar? Yes for Triangle.
Four node complete? Yes.
Five node complete or $K_5$? No! Why? Later.

Two to three nodes, bipartite? Yes.
Planar graphs.

A graph that can be drawn in the plane without edge crossings.

Planar? Yes for Triangle.
Four node complete? Yes.
Five node complete or $K_5$? No! Why? Later.

Two to three nodes, bipartite? Yes.
Three to three nodes, complete/bipartite or $K_{3,3}$. 
Planar graphs.

A graph that can be drawn in the plane without edge crossings.

Planar? Yes for Triangle.
Four node complete? Yes.
Five node complete or $K_5$? No! Why? Later.

Two to three nodes, bipartite? Yes.
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Planar graphs.

A graph that can be drawn in the plane without edge crossings.

Planar? Yes for Triangle.
Four node complete? Yes.
Five node complete or $K_5$? No! Why? Later.

Two to three nodes, bipartite? Yes.
Three to three nodes, complete/bipartite or $K_{3,3}$? No. Why? Later.
Euler’s Formula.

Faces: connected regions of the plane.

How many faces for triangle?

2 complete on four vertices or $K_4$?

4 bipartite, complete two/three or $K_2, 3$?

$v$ is number of vertices, $e$ is number of edges, $f$ is number of faces.

Euler’s Formula: Connected planar graph has $v + f = e + 2$.

Triangle: $3 + 2 = 3 + 2!$

$K_4$: $4 + 4 = 6 + 2!$

$K_2, 3$: $5 + 3 = 6 + 2!$

Examples = 3!

Proven! Not!!!!
Euler’s Formula.

Faces: connected regions of the plane.

Euler’s Formula: Connected planar graph has $v + f = e + 2$.

Examples:

- Triangle: $3 + 2 = 3 + 2$.
- $K_4$: $4 + 4 = 6 + 2$.
- $K_{2,3}$: $5 + 3 = 6 + 2$.

Proven! Not!!!!
Euler’s Formula.

Faces: connected regions of the plane.

How many faces for

Examples = 3!

Proven! Not!!!!
Euler’s Formula.

Faces: connected regions of the plane.

How many faces for triangle?

Examples = 3!

Proven! Not!!!!
Euler’s Formula.

Faces: connected regions of the plane.

How many faces for triangle? 2
Euler’s Formula.

Faces: connected regions of the plane.

How many faces for
triangle? 2
complete on four vertices or $K_4$?
Euler’s Formula.

Faces: connected regions of the plane.

How many faces for
triangle? 2
complete on four vertices or \( K_4 \)? 4
Euler’s Formula.

Faces: connected regions of the plane.

How many faces for
  triangle? 2
  complete on four vertices or $K_4$? 4
  bipartite, complete two/three or $K_{2,3}$?
Euler’s Formula.

Faces: connected regions of the plane.

How many faces for
- triangle? 2
- complete on four vertices or $K_4$? 4
- bipartite, complete two/three or $K_{2,3}$? 3

Examples = 3!
Proven! Not!!!!
Euler’s Formula.

Faces: connected regions of the plane.

How many faces for
- triangle? 2
- complete on four vertices or $K_4$? 4
- bipartite, complete two/three or $K_{2,3}$? 3

Examples = 3!  
Proven! Not!!!!
Euler’s Formula.

Faces: connected regions of the plane.

How many faces for
  - triangle? 2
  - complete on four vertices or $K_4$? 4
  - bipartite, complete two/three or $K_{2,3}$? 3

$v$ is number of vertices, $e$ is number of edges, $f$ is number of faces.
Euler’s Formula.

Faces: connected regions of the plane.

How many faces for
  triangle? 2
  complete on four vertices or $K_4$? 4
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Euler’s Formula.

Faces: connected regions of the plane.

How many faces for
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\( v \) is number of vertices, \( e \) is number of edges, \( f \) is number of faces.

**Euler’s Formula:** Connected planar graph has \( v + f = e + 2 \).

Triangle:
Euler’s Formula.

Faces: connected regions of the plane.

How many faces for

- triangle? 2
- complete on four vertices or $K_4$? 4
- bipartite, complete two/three or $K_{2,3}$? 3

$v$ is number of vertices, $e$ is number of edges, $f$ is number of faces.

**Euler’s Formula**: Connected planar graph has $v + f = e + 2$.

Triangle: $3 + 2 = 3 + 2!$
Euler’s Formula.

Faces: connected regions of the plane.

How many faces for
  triangle? 2
  complete on four vertices or $K_4$? 4
  bipartite, complete two/three or $K_{2,3}$? 3

$v$ is number of vertices, $e$ is number of edges, $f$ is number of faces.

**Euler’s Formula:** Connected planar graph has $v + f = e + 2$.

Triangle: $3 + 2 = 3 + 2!$

$K_4$: 
Euler’s Formula.

Faces: connected regions of the plane.

How many faces for
triangle? 2
complete on four vertices or $K_4$? 4
bipartite, complete two/three or $K_{2,3}$? 3

$v$ is number of vertices, $e$ is number of edges, $f$ is number of faces.

**Euler’s Formula:** Connected planar graph has $v + f = e + 2$.

Triangle: $3 + 2 = 3 + 2!$

$K_4$: $4 + 4 = 6 + 2!$
Euler’s Formula.

Faces: connected regions of the plane.

How many faces for
  triangle? 2
  complete on four vertices or $K_4$? 4
  bipartite, complete two/three or $K_{2,3}$? 3

$v$ is number of vertices, $e$ is number of edges, $f$ is number of faces.

**Euler’s Formula**: Connected planar graph has $v + f = e + 2$.

Triangle: $3 + 2 = 3 + 2$!
$K_4$: $4 + 4 = 6 + 2$!
$K_{2,3}$:
Euler’s Formula.

Faces: connected regions of the plane.

How many faces for
- triangle? 2
- complete on four vertices or $K_4$? 4
- bipartite, complete two/three or $K_{2,3}$? 3

$v$ is number of vertices, $e$ is number of edges, $f$ is number of faces.

**Euler’s Formula:** Connected planar graph has $v + f = e + 2$.

Triangle: $3 + 2 = 3 + 2!$
$K_4$: $4 + 4 = 6 + 2!$
$K_{2,3}$: $5 + 3 = 6 + 2!$
Euler’s Formula.

Faces: connected regions of the plane.

How many faces for
- triangle? 2
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Examples = 3!
Euler’s Formula.

Faces: connected regions of the plane.

How many faces for
- triangle? 2
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Examples = 3! Proven!
Euler’s Formula.

Faces: connected regions of the plane.

How many faces for

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Examples = 3! Proven! Not!!!!
Euler and Polyhedron.

Greeks knew formula for polyhedron.
Euler and Polyhedron.

Greeks knew formula for polyhedron.
Euler and Polyhedron.

Greeks knew formula for polyhedron.

Faces?

8 + 6 = 12 + 2.

Euler and Polyhedron.

Greeks knew formula for polyhedron.

Faces? 6. Edges?

Greeks couldn't prove it. Induction?

Polyhedron without holes ≡ Planar graphs.

Surround by sphere. Project from point inside polytope onto sphere. Sphere ≡ Plane!

Topologically. Euler proved formula thousands of years later!
Euler and Polyhedron.

Greeks knew formula for polyhedron.


Euler: Connected planar graph: $v + f = e + 2$.

Greeks couldn't prove it. Induction? Remove vertex for polyhedron?

Polyhedron without holes $\equiv$ Planar graphs. Surround by sphere. Project from point inside polytope onto sphere. Sphere $\equiv$ Plane! Topologically. Euler proved formula thousands of years later!
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Greeks couldn't prove it.  Induction?  Remove vertice for polyhedron?  Polyhedron without holes \( \equiv \) Planar graphs.

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Euler: Connected planar graph:
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Euler: Connected planar graph: \( v + f = e + 2 \).
\[ 8 + 6 = 12 + 2. \]
Euler and Polyhedron.

Greeks knew formula for polyhedron.


**Euler:** Connected planar graph: \( v + f = e + 2 \).

\[ 8 \text{ } + \text{ } 6 \text{ } = \text{ } 12 \text{ } + \text{ } 2. \]

Greeks couldn’t prove it.
Euler and Polyhedron.

Greeks knew formula for polyhedron.


**Euler:** Connected planar graph: \( v + f = e + 2 \).

\[ 8 + 6 = 12 + 2. \]

Greeks couldn’t prove it. Induction?
Euler and Polyhedron.

Greeks knew formula for polyhedron.


Euler: Connected planar graph: $v + f = e + 2$.

$8 + 6 = 12 + 2$.

Greeks couldn’t prove it. Induction? Remove vertex for polyhedron?
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Euler and Polyhedron.

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\[8 + 6 = 12 + 2.\]

Greeks couldn’t prove it. Induction? Remove vertex for polyhedron?

Polyhedron without holes
Euler and Polyhedron.

Greeks knew formula for polyhedron.


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\[ 8 + 6 = 12 + 2. \]

Greeks couldn’t prove it. Induction? Remove vertex for polyhedron?

Polyhedron without holes \( \equiv \).
Euler and Polyhedron.

Greeks knew formula for polyhedron.


**Euler:** Connected planar graph: \( v + f = e + 2 \).
\[ 8 + 6 = 12 + 2. \]

Greeks couldn’t prove it. Induction? Remove vertice for polyhedron?

Polyhedron without holes \( \equiv \) Planar graphs.
Euler and Polyhedron.

Greeks knew formula for polyhedron.

Edges? 12.  
Vertices? 8.

**Euler:** Connected planar graph: \( v + f = e + 2 \).
\[ 8 + 6 = 12 + 2. \]

Greeks couldn’t prove it. Induction? Remove vertex for polyhedron?

Polyhedron without holes \( \equiv \) Planar graphs.

Surround by sphere.
Euler and Polyhedron.

 Greeks knew formula for polyhedron.


 Euler: Connected planar graph:  \( v + f = e + 2 \).
  \( 8 + 6 = 12 + 2 \).

 Greeks couldn’t prove it. Induction? Remove vertex for polyhedron?

 Polyhedron without holes \( \equiv \) Planar graphs.
  Surround by sphere.
  Project from point inside polytope onto sphere.
Euler and Polyhedron.

Greeks knew formula for polyhedron.


**Euler: Connected planar graph:** \( v + f = e + 2 \).
\[ 8 + 6 = 12 + 2. \]

Greeks couldn’t prove it. Induction? Remove vertex for polyhedron?

Polyhedron without holes \( \equiv \) Planar graphs.

Surround by sphere.
Project from point inside polytope onto sphere.
Sphere
Euler and Polyhedron.

Greeks knew formula for polyhedron.


**Euler:** Connected planar graph: \( v + f = e + 2 \).
\[ 8 + 6 = 12 + 2. \]

Greeks couldn’t prove it. Induction? Remove vertex for polyhedron?

Polyhedron without holes \( \equiv \) Planar graphs.

Surround by sphere.
Project from point inside polytope onto sphere.
Sphere \( \equiv \) Plane!
Euler and Polyhedron.

Greeks knew formula for polyhedron.


Euler: Connected planar graph: \( v + f = e + 2 \).
\[ 8 + 6 = 12 + 2 \]

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Sphere \( \equiv \) Plane! Topologically.
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Euler proved formula thousands of years later!
Euler and planarity of $K_5$ and $K_{3,3}$
Euler and planarity of $K_5$ and $K_{3,3}$

Euler: $v + f = e + 2$ for connected planar graph.
Euler and planarity of $K_5$ and $K_{3,3}$

Euler: $v + f = e + 2$ for connected planar graph.

Each face is adjacent to at least three edges.
Euler and planarity of $K_5$ and $K_{3,3}$

Euler: $v + f = e + 2$ for connected planar graph.

Each face is adjacent to at least three edges. Face-edge adjacencies. $\geq 3f$
Euler: \( v + f = e + 2 \) for connected planar graph.

Each face is adjacent to at least three edges. face-edge adjacencies. \( \geq 3f \)
Each edge is adjacent to exactly two faces.

\[ 10 \neq 3(5) - 6 = 9 \Rightarrow K_5 \text{ is not planar!} \]

\[ 9 \leq 2(6) - 4 = 8 \Rightarrow K_{3,3} \text{ is not planar!} \]
Euler and planarity of $K_5$ and $K_{3,3}$

Euler: $v + f = e + 2$ for connected planar graph.

- Each face is adjacent to at least three edges. face-edge adjacencies. $\geq 3f$
- Each edge is adjacent to exactly two faces. face-edge adjacencies. $= 2e$

$K_5$:
- Edges? 10
- Vertices? 5

$10 \not\leq 3(5) - 6 = 9 \Rightarrow K_5$ is not planar.

$K_{3,3}$:
- Edges? 9
- Vertices? 6

$9 \leq 2(6) - 4 = 8 \Rightarrow K_{3,3}$ is planar!
Euler and planarity of $K_5$ and $K_{3,3}$

Euler: $v + f = e + 2$ for connected planar graph.

Each face is adjacent to at least three edges. face-edge adjacencies. $\geq 3f$

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$\implies 3f \leq 2e$
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Euler: $v + \frac{2}{3}e \geq e + 2$
Euler and planarity of $K_5$ and $K_{3,3}$

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Each edge is adjacent to exactly two faces. face-edge adjacencies. $= 2e$
$\implies 3f \leq 2e$

Euler: $v + \frac{2}{3}e \geq e + 2 \implies e \leq 3v - 6$
Euler and planarity of $K_5$ and $K_{3,3}$

Euler: $v + f = e + 2$ for connected planar graph.

- Each face is adjacent to at least three edges: face-edge adjacencies $\geq 3f$
- Each edge is adjacent to exactly two faces: face-edge adjacencies $= 2e$

$\implies 3f \leq 2e$

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$K_5$

$K_{3,3}$
Euler and planarity of $K_5$ and $K_{3,3}$

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$K_5$ Edges?

$K_{3,3}$ Edges?
Euler and planarity of $K_5$ and $K_{3,3}$

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Euler: $v + \frac{2}{3} e \geq e + 2 \implies e \leq 3v - 6$

$K_5$ Edges? $4 + 3 + 2 + 1$
Euler and planarity of $K_5$ and $K_{3,3}$

Euler: $v + f = e + 2$ for connected planar graph.

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Each edge is adjacent to exactly two faces. face-edge adjacencies. $= 2e$

$\implies 3f \leq 2e$

Euler: $v + \frac{2}{3}e \geq e + 2 \implies e \leq 3v - 6$

$K_5$ Edges? $4 + 3 + 2 + 1 = 10$. 

$K_{3,3}$ Edges? $6 + 3 + 3 + 1 = 13$. 

Sure! But no cycles that are triangles. Face is of length $\geq 4$. 

$4f \leq 2e$. 

Euler: $v + \frac{2}{3}e \geq e + 2 \implies e \leq 3v - 6$

$K_{3,3}$ is not planar! 

$K_5$ is not planar.
Euler and planarity of $K_5$ and $K_{3,3}$

Euler: $v + f = e + 2$ for connected planar graph.

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$\implies 3f \leq 2e$

Euler: $v + \frac{2}{3}e \geq e + 2 \implies e \leq 3v - 6$

$K_5$ Edges? $4 + 3 + 2 + 1 = 10$. Vertices?
Euler and planarity of $K_5$ and $K_{3,3}$

Euler: $v + f = e + 2$ for connected planar graph.

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Each edge is adjacent to exactly two faces. face-edge adjacencies. $= 2e$
\[\implies 3f \leq 2e\]

Euler: $v + \frac{5}{3} e \geq e + 2 \implies e \leq 3v - 6$

Euler and planarity of $K_5$ and $K_{3,3}$

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$10 \not\leq 3(5) - 6 = 9$. 
Euler and planarity of $K_5$ and $K_{3,3}$

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$10 \leq 3(5) - 6 = 9. \implies K_5$ is not planar.
Euler and planarity of $K_5$ and $K_{3,3}$

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$10 \leq 3(5) - 6 = 9$. $\implies K_5$ is not planar.

$K_{3,3}$?
Euler and planarity of $K_5$ and $K_{3,3}$

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Euler and planarity of $K_5$ and $K_{3,3}$

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$10 \not\leq 3(5) - 6 = 9. \implies K_5$ is not planar.

$K_{3,3}$? Edges? 9. Vertices. 6. $9 \not\leq 3(6) - 6$?
Euler and planarity of $K_5$ and $K_{3,3}$

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10 $\leq 3(5) - 6 = 9$. $\implies K_5$ is not planar.

$K_{3,3}$? Edges? 9. Vertices. 6. $9 \leq 3(6) - 6$? Sure!
Euler and planarity of $K_5$ and $K_{3,3}$

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$10 \leq 3(5) - 6 = 9$. $\implies K_5$ is not planar.

$K_{3,3}$? Edges? 9. Vertices. 6. $9 \leq 3(6) - 6$? Sure!
But no cycles that are triangles.
Euler and planarity of $K_5$ and $K_{3,3}$

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$K_{3,3}$? Edges? 9. Vertices. 6. $9 \leq 3(6) - 6$? Sure!
But no cycles that are triangles. Face is of length $\geq 4$. 
Euler and planarity of $K_5$ and $K_{3,3}$

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Euler: $v + \frac{2}{3}e \geq e + 2 \implies e \leq 3v - 6$

$10 \leq 3(5) - 6 = 9. \implies K_5$ is not planar.

$K_{3,3}$? Edges? 9. Vertices. 6. $9 \leq 3(6) - 6$? Sure!
But no cycles that are triangles. Face is of length $\geq 4$.
.... $4f \leq 2e$. 
**Euler and planarity of** $K_5$ **and** $K_{3,3}$

Euler: \( v + f = e + 2 \) for connected planar graph.

- Each face is adjacent to at least three edges. face-edge adjacencies. \( \geq 3f \)
- Each edge is adjacent to exactly two faces. face-edge adjacencies. \( = 2e \)

\[ 3f \leq 2e \]

Euler: \( v + \frac{2}{3}e \geq e + 2 \implies e \leq 3v - 6 \)

  - \( 10 \leq 3(5) - 6 = 9 \). \( \implies K_5 \) is not planar.

- $K_{3,3}$? Edges? 9. Vertices. 6. \( 9 \leq 3(6) - 6 \)? Sure!
- But no cycles that are triangles. Face is of length \( \geq 4 \).
  - \( 4f \leq 2e \)
  - Euler: \( v + \frac{1}{2}e \geq e + 2 \)
Euler and planarity of $K_5$ and $K_{3,3}$

Euler: $v + f = e + 2$ for connected planar graph.

- Each face is adjacent to at least three edges. face-edge adjacencies. $\geq 3f$
- Each edge is adjacent to exactly two faces. face-edge adjacencies. $= 2e$

$\implies 3f \leq 2e$

Euler: $v + \frac{2}{3}e \geq e + 2 \implies e \leq 3v - 6$

$10 \leq 3(5) - 6 = 9. \implies K_5$ is not planar.

$K_{3,3}$? Edges? 9. Vertices. 6. $9 \leq 3(6) - 6$? Sure!
But no cycles that are triangles. Face is of length $\geq 4$.
$\implies 4f \leq 2e.$
Euler: $v + \frac{1}{2}e \geq e + 2 \implies e \leq 2v - 4$
Euler and planarity of $K_5$ and $K_{3,3}$

Euler: $v + f = e + 2$ for connected planar graph.

Each face is adjacent to at least three edges. 

Each edge is adjacent to exactly two faces.

$\Rightarrow 3f \leq 2e$

Euler: $v + \frac{2}{3}e \geq e + 2 \implies e \leq 3v - 6$


$10 \leq 3(5) - 6 = 9. \implies K_5$ is not planar.

$K_{3,3}$? Edges? 9. Vertices. 6. $9 \leq 3(6) - 6$? Sure!

But no cycles that are triangles. Face is of length $\geq 4$.

$\ldots 4f \leq 2e$.

Euler: $v + \frac{1}{2}e \geq e + 2 \implies e \leq 2v - 4$
Euler and planarity of $K_5$ and $K_{3,3}$

Euler: $v + f = e + 2$ for connected planar graph.

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Each edge is adjacent to exactly two faces. face-edge adjacencies. $= 2e$

$\implies 3f \leq 2e$

Euler: $v + \frac{2}{3}e \geq e + 2 \implies e \leq 3v - 6$


$10 \leq 3(5) - 6 = 9. \implies K_5$ is not planar.

$K_{3,3}$? Edges? 9. Vertices. 6. $9 \leq 3(6) - 6$? Sure!

But no cycles that are triangles. Face is of length $\geq 4$.

$\implies 4f \leq 2e$.

Euler: $v + \frac{1}{2}e \geq e + 2 \implies e \leq 2v - 4$

$9 \leq 2(6) - 4.$
Euler and planarity of $K_5$ and $K_{3,3}$

Euler: $v + f = e + 2$ for connected planar graph.

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Each edge is adjacent to exactly two faces. face-edge adjacencies. $= 2e$
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Euler: $v + \frac{2}{3}e \geq e + 2 \implies e \leq 3v - 6$

$10 \not\leq 3(5) - 6 = 9. \implies K_5$ is not planar.

$K_{3,3}$? Edges? 9. Vertices. 6. $9 \leq 3(6) - 6$? Sure!
But no cycles that are triangles. Face is of length $\geq 4$.
$\implies 4f \leq 2e$.
Euler: $v + \frac{1}{2}e \geq e + 2 \implies e \leq 2v - 4$
$9 \not\leq 2(6) - 4. \implies K_{3,3}$ is not planar!
A tree is a connected acyclic graph.
A tree is a connected acyclic graph.

To tree or not to tree!
A tree is a connected acyclic graph.

To tree or not to tree!
A tree is a connected acyclic graph.

To tree or not to tree!

Yes.
A tree is a connected acyclic graph.

To tree or not to tree!

Yes. No.
A tree is a connected acyclic graph.

To tree or not to tree!

Yes. No. Yes.
A tree is a connected acyclic graph.

To tree or not to tree!

Yes. No. Yes. No.
A tree is a connected acyclic graph.

To tree or not to tree!

Yes. No. Yes. No. No.
A tree is a connected acyclic graph.

To tree or not to tree!


Faces?
A tree is a connected acyclic graph.

To tree or not to tree!

Yes. No. Yes. No. No.

Faces? 1.
A tree is a connected acyclic graph.

To tree or not to tree!

Yes. No. Yes. No. No.

Faces? 1. 2.
A tree is a connected acyclic graph.

To tree or not to tree!

Yes. No. Yes. No. No.

Faces? 1. 2. 1.
A tree is a connected acyclic graph.

To tree or not to tree!

Yes. No. Yes. No. No.

Faces? 1. 2. 1. 1.
A tree is a connected acyclic graph.

To tree or not to tree!

Yes. No. Yes. No. No.

Faces? 1. 2. 1. 1. 2.
A tree is a connected acyclic graph.

To tree or not to tree!

Yes. No. Yes. No. No.

Faces? 1. 2. 1. 1. 2.

Vertices/Edges.
A tree is a connected acyclic graph.

To tree or not to tree!

Yes. No. Yes. No. No.

Faces? 1. 2. 1. 1. 2.

Vertices/Edges. Notice: \( e = v - 1 \) for tree.
A tree is a connected acyclic graph.

To tree or not to tree!

Yes. No. Yes. No. No.

Faces? 1. 2. 1. 1. 2.
Vertices/Edges. Notice: $e = v - 1$ for tree.
A tree is a connected acyclic graph.

To tree or not to tree!

Yes. No. Yes. No. No.

Faces? 1. 2. 1. 1. 2.
Vertices/Edges. Notice: $e = v - 1$ for tree.

One face for trees!
A tree is a connected acyclic graph.

To tree or not to tree!

Yes. No. Yes. No. No.

Faces? 1. 2. 1. 1. 2.
Vertices/Edges. Notice: $e = v - 1$ for tree.

One face for trees!

Euler works for trees: $v + f = e + 2$. 
A tree is a connected acyclic graph.

To tree or not to tree!

Yes. No. Yes. No. No.

Faces? 1. 2. 1. 1. 2.

Vertices/Edges. Notice: $e = v - 1$ for tree.

One face for trees!

Euler works for trees: $v + f = e + 2$.
$v + 1 = v - 1 + 2$
A tree is a connected acyclic graph.

To tree or not to tree!

Yes. No. Yes. No. No.

Faces? 1. 2. 1. 1. 2.

Vertices/Edges. Notice: $e = v - 1$ for tree.

One face for trees!

Euler works for trees: $v + f = e + 2$. 
$v + 1 = v - 1 + 2$
Euler: Connected planar graph has \( v + f = e + 2 \).
Euler’s formula.

Euler: Connected planar graph has \( v + f = e + 2 \).

Proof sketch:
Euler: Connected planar graph has $v + f = e + 2$.

**Proof sketch:** Induction on $e$. 
Euler’s formula.

Euler: Connected planar graph has $v + f = e + 2$.

Proof sketch: Induction on $e$.
Base:

...
Euler’s formula.

Euler: Connected planar graph has $v + f = e + 2$.

Proof sketch: Induction on $e$.
Base: $e = 0$, 
Euler’s formula.

Euler: Connected planar graph has \( v + f = e + 2 \).

Proof sketch: Induction on \( e \).
Base: \( e = 0, v = f = 1 \).
Euler’s formula.

Euler: Connected planar graph has $v + f = e + 2$.

Proof sketch: Induction on $e$.
Base: $e = 0, v = f = 1$. $p(0)$ (base case) holds
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![Diagram of a graph](attachment:image)

Outer face.

Joins two faces.
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    Find a cycle. Remove edge.
    Joins two faces.
New graph: \( \nu \)-vertices.
Euler’s formula.

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\[
\begin{align*}
   \text{Outer face.} \\
   \text{Joins two faces.} \\
   \text{New graph: } v \text{-vertices. } e - 1 \text{ edges.}
\end{align*}
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**Euler’s formula.**

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![Graph](image)

Outer face.

Joins two faces.

New graph: \( v \)-vertices. \( e - 1 \) edges. \( f - 1 \) faces.
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  If it is a tree. Done.
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    Find a cycle. Remove edge.
    Joins two faces.
    New graph: \( v \)-vertices. \( e - 1 \) edges. \( f - 1 \) faces. Planar.
Euler’s formula.

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\( v + (f-1) = (e-1) + 2 \) by induction hypothesis for a smaller graph with \( e-1 \) edges.
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New graph: $v$-vertices. $e - 1$ edges. $f - 1$ faces. Planar.
$v + (f - 1) = (e - 1) + 2$ by induction hypothesis for a smaller graph with $e - 1$ edges.
Therefore $v + f = e + 2$. 
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Therefore $v + f = e + 2$. □
Graph Coloring.

Given $G = (V, E)$, a coloring of a $G$ assigns colors to vertices $V$ where for each edge the endpoints have different colors.
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![Graph Coloring Diagrams]

Notice that the last one has only three colors, fewer than the number of vertices and fewer than the maximum degree node.

Interesting things to do:

Algorithm!
Given $G = (V, E)$, a coloring of a $G$ assigns colors to vertices $V$ where for each edge the endpoints have different colors.

Notice that the last one, has one three colors.
Graph Coloring.

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Notice that the last one, has one three colors. Fewer colors than number of vertices.
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Given $G = (V, E)$, a coloring of a $G$ assigns colors to vertices $V$ where for each edge the endpoints have different colors.

Notice that the last one, has one three colors. Fewer colors than number of vertices. Fewer colors than max degree node.
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Interesting things to do. Algorithm!
Planar graphs and maps.

Planar graph coloring $\equiv$ map coloring.
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Four color theorem is about planar graphs!
**Theorem:** Every planar graph can be colored with six colors.
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Proof:
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Recall: $e \leq 3v - 6$ for any planar graph.
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  From Euler's Formula.
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Total degree: \(2e\)
Theorem: Every planar graph can be colored with six colors.

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Total degree: $2e$
Average degree: $\leq \frac{2e}{v}$
**Theorem:** Every planar graph can be colored with six colors.

**Proof:**
Recall: \( e \leq 3v - 6 \) for any planar graph.
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Total degree: \( 2e \)
Average degree: \( \leq \frac{2e}{v} \leq \frac{2(3v-6)}{v} \)
**Theorem:** Every planar graph can be colored with six colors.

**Proof:**
Recall: $e \leq 3v - 6$ for any planar graph.
   - From Euler’s Formula.

Total degree: $2e$
Average degree: \[ \leq \frac{2e}{v} \leq \frac{2(3v-6)}{v} \leq 6 - \frac{12}{v}. \]
**Theorem:** Every planar graph can be colored with six colors.

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There exists a vertex with degree \( < 6 \)
**Theorem:** Every planar graph can be colored with six colors.

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Recall: $e \leq 3v - 6$ for any planar graph.
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Total degree: $2e$
Average degree: $\leq \frac{2e}{v} \leq \frac{2(3v-6)}{v} \leq 6 - \frac{12}{v}$.

There exists a vertex with degree $< 6$ or at most 5.
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Remove vertex $v$ of degree at most 5.
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   Inductively color remaining graph.
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   Color is available for $v$ since only five neighbors...
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\( \square \)
Theorem: Every planar graph can be colored with five colors.
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Preliminary Observation: Connected components of vertices with two colors in a legal coloring can switch colors.
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Proof:
Again with the degree 5 vertex.
Theorem: Every planar graph can be colored with five colors.

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Again with the degree 5 vertex. Again recurse.
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m Assume neighbors are colored all differently.

What color is it?
Must be blue or green to be on that path.
Must be red or orange to be on that path.
Contradiction.
Can recolor one of the neighbors.
And recolor "center" vertex.
Theorem: Every planar graph can be colored with five colors.

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m Assume neighbors are colored all differently.
Otherwise done.
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Preliminary Observation: Connected components of vertices with two colors in a legal coloring can switch colors.

Proof:
Again with the degree 5 vertex. Again recurse. Assume neighbors are colored all differently. Otherwise done. Switch green to blue in component.

Planar. ⇒ paths intersect at a vertex!

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Planar. $\implies$ paths intersect at a vertex!
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Four Color Theorem

Theorem:
Any planar graph can be colored with four colors.

Proof:
Not Today!
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