Today (and tomorrow, and Wednesday)

Review: what have we done in class?

Future classes: where do you go next?

Applications: how is the stuff you learned in 70 useful in the real world?

Research frontiers: what are people in academia working on (related to 70) right now?

Gigs: interesting stuff with material for fun and practice!

**Announcement:** No scantron HKN surveys now (or ever again!). Everything’s online now. You should have received an email about this sometime in the previous week or two.
Propositional Logic

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Sanity check: Why is “∀q ∈ ℝ : |q| ≥ q” a statement but “∀x ∈ ℝ : xy = 0” not a statement?
Combining Statements

Boolean operators: and (\(\land\)), or (\(\lor\)), not (\(\neg\) or a line over your expression, e.g. \(\overline{x}\)), conditional (\(\implies\)), biconditional (\(\iff\)).
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Boolean operators: and ($\wedge$), or ($\vee$), not ($\neg$ or a line over your expression, e.g. $\bar{x}$), conditional ($\implies$), biconditional ($\iff$).

Prove things are equivalent by simplifying one (or both) sides, or with truth tables.

Example (SU14 MT1): True or false?

$P = (Q = R)$

Intuitive method: guess whether or not this seems right or not.

If $P$ true? Then $Q$ true, then $R$ true. If $P$ not true, then both sides reduce to $Q = R$ so they're equivalent.

Symbolic manipulation approach:

$P = (Q = R)$

$P \neg (Q = R)$

$P \neg (Q \neg R)$

$(P \neg Q) \neg R$
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Disprove things with counterexamples.
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Example (SU14 MT1): True or false? \(P \implies (Q \implies R) \equiv (P \land Q) \implies R\).
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\]
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Example (SU14 MT1): True or false? \(P \rightarrow (Q \rightarrow R) \equiv (P \land Q) \rightarrow R\).

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P \rightarrow (Q \rightarrow R) \equiv \overline{P} \lor (Q \rightarrow R) \equiv \overline{P} \lor (\overline{Q} \lor R) \equiv (\overline{P} \lor \overline{Q}) \lor R
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Boolean logic encompasses a lot of computation...
Applications: Circuits

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How big do circuits need to be in order to compute some function that you’re interested in? Circuit lower bounds. Hard problem. Lots of research going on about this. Absolute lower bounds that don’t depend on unproven assumptions are pretty primitive (think lower bounds for computing whether you have an even number of 1s as input).
Motivating example: $\exists x, y, z : (x \lor y \lor \overline{z}) \land (\overline{x} \lor y \lor z)$?

Yes, let $x$ be true, $y$ be true, and $z$ be anything.

Generally: given some Boolean formula... Is there some assignment of variables that makes it true? Do all assignments of variables make it true?

Satisfiability and tautology problems. Also well studied!

Satisfiability in the general case actually models almost any computation we care about.

Hard to solve in the general case though.

CS170, CS172 for info on why this is hard.

Sometimes we can estimate. Or we can satisfy some portion of the formula but not all.

CS174.

What's the probability that some formula has a satisfying assignment? Counting/probabilistic arguments. Interesting results. Phase transitions.
Motivating example: \( \exists x, y, z : (x \vee y \vee \bar{z}) \land (\bar{x} \vee y \vee z) \)? Yes, let \( x \) be true, \( y \) be true, and \( z \) be anything.
Applications: Satisfiability

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Proofs: Techniques

Direct proof. Just go and prove it!
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Contradiction. Suppose that what you’re trying to prove is wrong.
Prove that the universe implodes.
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Casewise. Split into cases. Make sure one of those cases always applies!
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Induction. Start from base case and expand to entire (countable) set with an inductive step.
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These are all techniques you can compose!
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How to prove the claim? Combinatorial proof. At the sandwich shop. They have $n$ sandwiches. I want to buy $k$, and eat one right now. How many ways?
Claim: for $0 \leq k \leq n$, $k \binom{n}{k} = n \binom{n-1}{k-1}$.

How to prove the claim? Combinatorial proof. At the sandwich shop. They have $n$ sandwiches. I want to buy $k$, and eat one right now. How many ways?

Answer 1: pick $k$ sandwiches to buy: $\binom{n}{k}$ ways. Pick one of them to eat: $k$ ways. Total: $k \binom{n}{k}$ ways.
**Claim**: for $0 \leq k \leq n$, $k \binom{n}{k} = n \binom{n-1}{k-1}$.

How to prove the claim? Combinatorial proof. At the sandwich shop. They have $n$ sandwiches. I want to buy $k$, and eat one right now. How many ways?

Answer 1: pick $k$ sandwiches to buy: $\binom{n}{k}$ ways. Pick one of them to eat: $k$ ways. Total: $k \binom{n}{k}$ ways.

Answer 2: pick sandwich to eat right now first: $n$ ways. Now pick $k-1$ sandwiches out of the remaining $n-1$ at the shop to take home: $\binom{n-1}{k-1}$ ways. Total: $n \binom{n-1}{k-1}$ ways.
Claim: for $0 \leq k \leq n$, $k\binom{n}{k} = n\binom{n-1}{k-1}$.

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Answer 2: pick sandwich to eat right now first: $n$ ways. Now pick $k-1$ sandwiches out of the remaining $n-1$ at the shop to take home: $\binom{n-1}{k-1}$ ways. Total: $n\binom{n-1}{k-1}$ ways.

Two quantities have to be the same. So we have proved the claim.
Example: FLT

**Fermat’s little theorem:** For all $p$ prime, $p|(a^p - a)$. 

**Proof:** by induction on $a$.

Base case: obviously $p$ must divide $0 = (0^p - 0) = (1^p - 1)$.

Suppose for induction that $p|a^p(a)$. It suffices to show that $p|(a + 1)^p$. Expand the first term (binomial theorem):

\[
(a + 1)^p = \sum_{k=0}^{p} \binom{p}{k} a^k = 1 + a^p + \sum_{k=1}^{p-1} \binom{p}{k} a^k.
\]

Notice that since $p$ is prime, $p|\binom{p}{k}$ for $k \geq 2$. Why?

From previous page:

\[
\binom{p}{k} = \frac{p(p-1)(p-2)\cdots(p-k+1)}{k!}.
\]

We know $p$ and $k$ have no common factors since $p$ is prime. So $p|\binom{p}{k}$.

Also $a^p(a) \equiv 0 \pmod{p}$ by inductive hypothesis.

So $(a + 1)^p = 1 + a^p + \sum_{k=1}^{p-1} \binom{p}{k} a^k \equiv 1 + a + 0 \pmod{p}$ as desired.
Fermat’s little theorem: For all $p$ prime, $p | (a^p - a)$.

Proof: by induction on $a$. 
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Proof: by induction on $a$. Base case: obviously $p$ must divide $0 = (0^p - 0) = (1^p - 1)$.

Suppose for induction that $p|(a^p - a)$. It suffices to show that $p|((a+1)^p - (a+1))$. 
Fermat’s little theorem: For all $p$ prime, $p|a^p - a$.

Proof: by induction on $a$. Base case: obviously $p$ must divide $0 = (0^p - 0) = (1^p - 1)$.

Suppose for induction that $p|a^p - a$. It suffices to show that $p|((a + 1)^p - (a + 1))$. Expand the first term (binomial theorem): $(a + 1)^p$
**Example: FLT**

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Suppose for induction that $p | (a^p - a)$. It suffices to show that $p | ((a + 1)^p - (a + 1))$. Expand the first term (binomial theorem): $(a + 1)^p = \sum_{k=0}^{p} \binom{p}{k} a^k = 1 + a^p + \sum_{k=1}^{p-1} \binom{p}{k} a^k.$
Fermat’s little theorem: For all \( p \) prime, \( p \mid (a^p - a) \).

Proof: by induction on \( a \). Base case: obviously \( p \) must divide 
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(a + 1)^p = \sum_{k=0}^{p} \binom{p}{k} a^k = 1 + a^p + \sum_{k=1}^{p-1} \binom{p}{k} a^k.
\]

Notice that since \( p \) is prime, \( p \mid \binom{p}{k} \) for \( k \in (0, p) \). Why?
Fermat’s little theorem: For all \( p \) prime, \( p|{(a^p - a)} \).

Proof: by induction on \( a \). Base case: obviously \( p \) must divide \( 0 = (0^p - 0) = (1^p - 1) \).

Suppose for induction that \( p|{(a^p - a)} \). It suffices to show that \( p|{( (a + 1)^p - (a + 1))} \). Expand the first term (binomial theorem):
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(a + 1)^p = \sum_{k=0}^{p} \binom{p}{k} a^k = 1 + a^p + \sum_{k=1}^{p-1} \binom{p}{k} a^k.
\]

Notice that since \( p \) is prime, \( p|{(\binom{p}{k})} \) for \( k \in (0, p) \). Why? From previous page: \( k\binom{p}{k} = p\binom{p-1}{k-1} \). We know \( p \) and \( k \) have no common factors since \( p \) is prime. So \( p|{(\binom{p}{k})} \).
Fermat’s little theorem: For all $p$ prime, $p|(a^p - a)$.

Proof: by induction on $a$. Base case: obviously $p$ must divide $0 = (0^p - 0) = (1^p - 1)$.

Suppose for induction that $p|(a^p - a)$. It suffices to show that $p|((a + 1)^p - (a + 1))$. Expand the first term (binomial theorem):

$$(a + 1)^p = \sum_{k=0}^{p} \binom{p}{k} a^k = 1 + a^p + \sum_{k=1}^{p-1} \binom{p}{k} a^k.$$

Notice that since $p$ is prime, $p|\binom{p}{k}$ for $k \in (0, p)$. Why? From previous page: $k\binom{p}{k} = p\binom{p-1}{k-1}$. We know $p$ and $k$ have no common factors since $p$ is prime. So $p|\binom{p}{k}$.

Also $a^p \equiv a \pmod{p}$ by inductive hypothesis.

So

$$(a + 1)^p - (a + 1) = 1 + a^p + \sum_{k=1}^{p-1} \binom{p}{k} a^k - (a + 1).$$
Example: FLT

Fermat’s little theorem: For all \( p \) prime, \( p | (a^p - a) \).

Proof: by induction on \( a \). Base case: obviously \( p \) must divide 
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&\equiv 1 + a + 0 - (a + 1) \pmod{p} \\
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\end{align*}$$

as desired.
Graphs

\[ G = (V, E). \] Collection of vertices (or nodes) and edges = pairs of vertices.
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(remember that a Markov chain represented by a strongly connected graph is irreducible).
Aside: Interesting Applications of Graphs

Web hyperlinks and social networks. Meshes in simulations and scientific computing.¹

Maps and grids. Games.

Finding paths in graphs is really useful. How does Google maps find a route to your destination? Finding paths in graphs! CS170, CS188.

If all degrees are even: **Eulerian tour** - can walk around the graph so we touch every edge exactly once and return to where we started.
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What about if we say that we want to touch every vertex? Interesting question... and hard. **CS170.**
Complete graphs.
Complete graphs.

**K**\textsuperscript{n} graphs. How many edges? \((n^2)\).
Complete graphs.

\[ K_n \]

\[ n \text{ vertices. How many edges?} \]

\[ \binom{n}{2} \]

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Complete graphs.

$K_n$. $n$ vertices. How many edges? $\binom{n}{2}$. 
Complete graphs.

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(INCLUDE) INCLUDE

ALL THE EDGES
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No odd length cycles.
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Can represent matchings. Remember stable matchings problem from MT1?

No odd length cycles. Random walk on a bipartite graph is periodic.
How do you define a tree?

- Connected acyclic graph.
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Which definition is correct? All of them are equivalent. Good practice exercise: prove it!
Trees are really useful!

Data structures! Binary search trees, heaps, red-black trees, B-trees, etc. Great for storing data. CS61B.
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Spawn trees. Represent function calls, etc.

Decision trees. Every vertex represents a decision you can make. \texttt{CS188}
Can you draw the graph on paper so that no edges cross? If so, it’s a planar graph.

Euler’s formula: $v + f = e + 2$. (how did we prove this? Induction on $e$.

Remove cycle by removing edge.)

Four color theorem: any planar graph can be colored with four colors so that no edge is monochromatic (same color on both endpoints). You can color a map with four colors. Proof? 400 pages long. Too long for this course... or the exam.

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For instance: if a program I’m trying to compile that looks like this:

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How many registers (memory) do I need to run this program? Draw a graph and try to approximate the optimal coloring! Each color is a register.

Register optimization! Touched on in CS164.
Hypercubes. $G = \{V, E\}$. $V = \{0, 1\}^n$.
$E = \{(u, v)|u$ and $v$ differ by exactly one bit position\}. 
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Shortest path from $u$ to $v$? How many bits do they differ in? “hamming distance”. Each edge traversal is a bit flip.

---

2. [https://blog.ethereum.org/2014/10/21/scalability-part-2-hypercubes/](https://blog.ethereum.org/2014/10/21/scalability-part-2-hypercubes/)
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Ease of routing and difficulty to cut make hypercubes really useful for distributed systems. Hypercube topology to be very common in supercomputers: Intel iPSC, nCube.

Now being used for routing messages in the Ethereum network.²

²https://blog.ethereum.org/2014/10/21/scalability-part-2-hypercubes/

Expected number of edges? $\binom{n}{2}p$. We used this for probabilistic method proofs.

Random walks on undirected graphs. Take a graph and turn it into a Markov chain. Vertices are states. Transition to any neighboring state with uniform probability.

Irreducible if graph is connected. Aperiodic if graph isn’t bipartite.

Stationary distribution? $\deg(v) = 2\left\lceil \frac{E}{j} \right\rceil$.

Cover time? bounded by $4\left\lceil \frac{V}{j} \right\rceil$. 

More review on this tomorrow.
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Want to match men and women in a stable manner. No rogue couples (people who would both rather be with each other than with their current partners).
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Each day: Each man proposes to the highest woman on his list who hasn’t rejected him yet. Each woman rejects all but the best, whom she “keeps on a string”.

Improvement lemma: If a woman has a man on a string: any future man who she has on a string is going to be at least as good.

TMA produces male optimal stable matching! Male optimal = best pairing for men among all possible matchings.

Theorem: male optimal = female pessimal.


Used in hospital residency matching systems. Matching in general is a well studied problem. Used in programs like kidney exchanges.
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Polling $k$ random people on the street in a city of population $n$? Order doesn’t matter, with replacement.

Making a homework by selecting $k$ problems from a set of $n$? Order matters, without replacement.

Answering $n$ multiple-choice questions, each with $k$ options?
Counting

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And so on!
Uncountability

What’s countable and what’s not?

- Finite bitstrings. **countable**
- Programs that print "CS70". **countable**
- Programs that loop infinitely. **countable**
- Finite sets of countable sets. **countable**
- Pairs of real numbers. **uncountable**
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So halting is undecidable!
Questions?