Markov Chains II

CS70 Summer 2016 - Lecture 6C

David Dinh
27 July 2016

UC Berkeley
Agenda

Classification of MC states
Aperiodicity, irreducibility, ergodicity
Convergence, limiting and stationary distributions

Reference for this lecture: Ch. 7 of Mitzenmacher and Upfal, ”Probability and Computing”
Markov Chain Properties
State $i$ is accessible from $j$ if there is some chance that, if I’m at $j$ at some timestep, I’ll end up at state $i$ some time later.
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If $j$ is accessible from $i$ and $i$ is accessible from $j$, then they are said to “communicate”.
Accessibility and Communication

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Another way of looking at it: directed connectivity.
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If $j$ is accessible from $i$ and $i$ is accessible from $j$, then they are said to “communicate”.

Another way of looking at it: directed connectivity. $i$ communicates with $j$: exists path from $i$ to $j$ in the graph corresponding to the chain.
Accessibility and Communication: Example

Is 1 accessible from 2? No.

Is 2 accessible from 1? Yes.

Do 1 and 2 communicate? No.

Is 2 accessible from 3? Yes.

Is 3 accessible from 2? Yes.

Do 1 and 2 communicate? Yes.
Accessibility and Communication: Example

Is 1 accessible from 2? No

Is 2 accessible from 1? Yes

Do 1 and 2 communicate? No

Is 2 accessible from 3? Yes

Is 3 accessible from 2? Yes

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Is 2 accessible from 3? Yes. Is 3 accessible from 2? Yes. Do 1 and 2 communicate? Yes.
Irreducible Markov chain: every state communicates with every other state.
Irreducibility

**Irreducible** Markov chain: every state communicates with every other state.

Or: graph representation is strongly connected.
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Irreducible.

Not irreducible.
Recurrent States

Let’s say we’re at a state $i$. Do we ever return to it again?

Is state 1 recurrent? No!
Recurrent States

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Let $r_{i,j}^t$ denote the probability that we first hit state $j$ in $t$ steps, starting from state $i$. 

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Is state 1 recurrent? *No!*
A Theorem

Suppose we are dealing with a finite MC. Then:

• There is at least one recurrent state.
• For any recurrent state $i$, the expected hitting time $h_i$; $i$ if we start from $i$ is finite.

Proof: (first part) Consider a non-recurrent state. If we start at that timestep, there is a nonzero probability that we will never see it again. Then if we start from that state and do an infinite number of timesteps, the probability that we see that state infinitely many times is zero.

Start anywhere on the MC and do an infinite number of timesteps. Since the MC is finite, some step must appear infinitely many times. So, that step must be recurrent.
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Aperiodicity

Intuition: Suppose we’re in one of these states at some timestep. Then we can never return to it an odd number of timesteps later. To capture this intuition: state $j$ is periodic if there exists some integer $\Delta > 1$ such that $P_{s,j;j} = Pr[X_{t+s} = j | X_t = j] = 0$ unless $\Delta$ divides $s$.

A Markov chain is said to be periodic if any of its states is periodic. Opposite of periodic: aperiodic.
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Opposite of periodic: aperiodic.
Aperiodicity of Irreducible Chains - Another Definition

**Theorem:** Assume that the MC is irreducible.

\[ d(j) := \gcd \{ f_i \} > 0 \quad j > 0 \]

has the same value for all states \( i \).

**Proof:** See Lecture note 18.

**Definition:**

If \( d(j) = 1 \), the Markov chain is said to be aperiodic. Otherwise, it is periodic with period \( d(j) \).

Are the definitions the same?

Yes.

On timesteps \( s \) that are not multiples of \( d(j) \), \( P_{s,j} \) is zero.

---

\(^1\)gcd = greatest common divisor.
Theorem: Assume that the MC is irreducible. Then

\[ d(j) := \text{g.c.d.}\{s > 0 \mid P_{j,j}^s > 0\} \]

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Ergodicity

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“Ludwig Boltzmann needed a word to express the idea that if you took an isolated system at constant energy and let it run, any one trajectory, continued long enough, would be representative of the system as a whole. Being a highly-educated nineteenth century German-speaker, Boltzmann knew far too much ancient Greek, so he called this the “ergodic property”, from ergon “energy, work” and hodos “way, path.” The name stuck.” (Advanced Data Analysis from an Elementary Point of View by Shalizi, pg. 479)
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**Theorem:** A finite, irreducible, aperiodic Markov chain is ergodic.
Stationary and Limiting Distributions
Consider the driving exam MC again.
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Once we pass the test (state 4), we’re done forever. We never leave state 4.
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If our distribution is $[0 \ 0 \ 0 \ 1]$; distribution is unchanged over a timestep.
Or how about the two-cycle?

![Diagram of a two-cycle model]

What if our distribution is 

\[
\begin{bmatrix}
0 & \frac{1}{2} \\
\frac{1}{2} & 0
\end{bmatrix}
\]

Does it change with timesteps?

No!
Or how about the two-cycle?

What if our distribution is \([0.5 \ 0.5]\)? Does it change with timesteps?
Or how about the two-cycle?

What if our distribution is $[0.5 \ 0.5]$? Does it change with timesteps? No!
A distribution $\pi$ over states in a Markov chain is said to be a stationary distribution (a.k.a. an invariant or equilibrium distribution) if $\pi = \pi P$. 
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To find stationary distribution: solve $\pi P = \pi$ ("balance equations")
An Example

These equations are redundant! Add equation equation:

\[ 1 + 2 = 1. \]

Solves to:

\[ P = \begin{bmatrix} 1-a & a \\ b & 1-b \end{bmatrix}. \]
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\iff \pi(1)(1 - a) + \pi_2 b = \pi_1 \text{ and }
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Add equation equation:

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Solves to:

\[ \pi = \begin{bmatrix} b \\ a + b \end{bmatrix} : \]

\[
\begin{align*}
\pi P &= \pi \\
\iff& \quad [\pi_1, \pi_2] \begin{bmatrix} 1 - a & a \\ b & 1 - b \end{bmatrix} = [\pi_1, \pi_2] \\
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\pi P = \pi \Leftrightarrow [\pi_1, \pi_2] \begin{bmatrix} 1 - a & a \\ b & 1 - b \end{bmatrix} = [\pi_1, \pi_2]
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Solves to:
\[ \pi P = \pi \quad \Rightarrow \quad [\pi_1, \pi_2] \begin{bmatrix} 1 - a & a \\ b & 1 - b \end{bmatrix} = [\pi_1, \pi_2] \]

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\[ \Rightarrow \quad \pi_1 a = \pi_2 b. \]

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Solves to:

\[ \pi = \begin{bmatrix} \frac{b}{a+b}, & \frac{a}{a+b} \end{bmatrix}. \]
Another Example

\[ P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \]

So:

\[ 1 = 1 \]
\[ 2 = 2 \]

Every distribution is invariant for this Markov chain. This is obvious, since \( X_n = X_0 \) for all \( n \). Hence, \( \Pr[X_n = i] = \Pr[X_0 = i] \), \( \forall (i; n) \).
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So:

$\pi_1 = \pi_1$ and $\pi_2 = \pi_2$. 

Hence, $Pr[X_n = i] = Pr[X_0 = i], \forall (i; n)$.
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Every distribution is invariant for this Markov chain. This is obvious, since \( X_n = X_0 \) for all \( n \). Hence, \( Pr[X_n = i] = Pr[X_0 = i], \forall (i, n) \).
Suppose we are given a finite, irreducible, aperiodic Markov chain. Then:

• There is a unique stationary distribution.

• For all $j, i$, the limit $\lim_{t \to \infty} P_{t,j,i}$ exists and is independent of $j$.

• $\pi_i = \lim_{t \to \infty} P_{t,j,i} = 1 = \pi_i$.

Proof: really long and messy, see note 18 or Ch. 7 of MU. (we won’t expect you to know this).
Main Theorem

Suppose we are given a finite, irreducible, aperiodic Markov chain. Then:

• There is a unique stationary distribution $\pi$. 

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Suppose we are given a finite, irreducible, aperiodic Markov chain. Then:

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Proof: really long and messy, see note 18 or Ch. 7 of MU. (we won't expect you to know this).
Main Theorem

Suppose we are given a finite, irreducible, aperiodic Markov chain. Then:

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It turns out that the convergence of the limiting distribution to the stationary distribution corresponds to a nice result from linear algebra: if you multiply a random vector by a matrix a lot of times, the result will converge towards an eigenvector (specifically, one corresponding to the highest eigenvalue) w.h.p.

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Connections between Linear Algebra and Markov Chains

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(No, you do not need to know this for the midterms and the homeworks).
The Gambler’s Ruin


You win when you get all your friend’s money. You lose when your friend gets all of yours.

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What if you and your friend are willing to bet different amounts?
Suppose you have $l_1$ dollars and your friend has $l_2$. Express as above Markov chain.
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States $-l_1, l_2$ are recurrent; all others are transient. What is the probability that you win (i.e. you hit state $l_2$ before $l_1$)?
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Let $P_i^t$ be the probability that you’re at state $i$ after $t$ timesteps.
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Denote amount of money you have at timestep $t$ as $W_t$. 

What's the expected amount of money you have after a single step? 0.

What's the expected gain after $t$ steps, $E[W_t]$? 0, by induction.

So:

$$E[W_t] = \sum_{i=0}^{\infty} \frac{1}{2}^i = 0$$

$$\lim_{t \to 1} E[W_t] = \frac{1}{2} \frac{1}{1 - \frac{1}{2}} = 0$$

Solve:

$$q = \frac{1}{2} \cdot \frac{1}{1 + \frac{1}{2}}$$

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Random Walks
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Suppose I give you a connected graph and you walk around on it randomly.

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Is it irreducible? Yes, if it’s connected.
**Theorem:** A random walk on an undirected, connected graph is aperiodic if and only if the graph is not bipartite.
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**Proof:** Suppose graph is bipartite. Then if I start at a node I can never go back in an odd number of timesteps. So random walk is periodic.

Conversely, suppose graph is not bipartite. Then there's an odd cycle (lecture 6). So we have a path of odd length from any node to itself. Then there exists an \( n' \) such that for all \( n \), I can go from my start node back to itself in \( n \) timesteps. Why?

If \( n \) is even: just go to the next node and back \( n = 2 \) times.

If \( n \) is odd: Go to some node in cycle (graph is connected). Traverse cycle. Go back. Going to node and back takes even number of timesteps. Traversing cycle takes odd number of timesteps. Total number of timesteps: odd.

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$$\pi_v = \sum_{u \in N(v)} \frac{d(u)}{2|E|} \frac{1}{d(u)} = \frac{d(v)}{2|E|}.$$ 

So $\pi$ solves the balance equations, so it’s stationary.
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Since $v \in N(u)$: $h_{v,u} < 2|E|$
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So: $4|E||V|$ is an upper bound on the cover time.
Application: PageRank

Idea: web search should give you results ordered in such a way that you're more likely to stumble on the top result than the lower results when browsing the web.

Assume you click links on webpages randomly forever. How often are you going to run into a webpage?

Model with a random walk on a directed graph. At each webpage:
- click random link.

Want to find the stationary distribution of this walk.

Problem: graph isn't strongly connected.

Solution: with small probability, go to a random website instead of clicking a link.

MC is irreducible and aperiodic, so its limiting distribution must be the unique stationary distribution.

Find the limiting distribution by solving an eigenvalue problem!

(Math 128B, Math 221)
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Find the limiting distribution by solving an eigenvalue problem! (Math 128B, Math 221)
Gig: Random Text