Quiz is out! Due: Friday at noon.

What are Markov Chains? State machine and matrix representations.

Hitting Time
Suppose we flip a coin until we get a three heads in a row. How many coin flips should we expect to do?
Motivation

Suppose we flip a coin until we get a three heads in a row. How many coin flips should we expect to do?

Drunkard on an arbitrary graph (remember HW?). When does the drunkard come home?
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Try solving directly? Problem: conditioning gets really messy.

Need some way to express state.

Solution: Markov chains!
Intuition

A finite Markov chain consists of states, transition probabilities between states, and an initial distribution.

State: where are you now?
Transition probability: From where you are, where do you go next?
Initial distribution: how do you start?

Markov chains are memoryless - they don't remember anything other than what state they are.
A finite Markov chain consists of states, *transition probabilities* between states, and an *initial distribution*.
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State: where are you now?

Transition probability: From where you are, where do you go next?

Initial distribution: how do you start?

Markov chains are *memoryless* - they don’t remember anything other than what state they are.
Formally Speaking...

A finite set of states: $X = f_1; 2; \ldots; Kg$

Initial probability distribution $0$ on $X$:

$0(i) \geq 0$ and $\sum_i 0(i) = 1$

Transition probabilities $P(i; j)$ for $i, j \in X$:

$P(i; j) \geq 0$ and $\sum_j P(i; j) = 1$ if $X_n; \ldots; X_1 = i$.

- $\Pr[X_0 = i] = 0(i); i \in X$ (initial distribution)
- $\Pr[X_{n+1} = j | X_0; \ldots; X_n = i] = P(i; j); i, j \in X$.
Formally Speaking...

A finite set of states: $\mathcal{X} = \{1, 2, \ldots, K\}$
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A initial probability distribution $\pi_0$ on $\mathcal{X}$: $\pi_0(i) \geq 0$, $\sum_i \pi_0(i) = 1$
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Transition probabilities: $P(i, j)$ for $i, j \in \mathcal{X}$
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Transition probabilities: \( P(i,j) \) for \( i,j \in \mathcal{X} \)

\[ P(i,j) \geq 0, \forall i,j; \]
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- $P(i, j) \geq 0$, $\forall i, j$; $\sum_j P(i, j) = 1$, $\forall i$

$\{X_n, n \geq 0\}$ is defined so that:
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$\{X_n, n \geq 0\}$ is defined so that:

- $Pr[X_0 = i] = \pi_0(i), i \in \mathcal{X}$
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A initial probability distribution \( \pi_0 \) on \( \mathcal{X} \) : \( \pi_0(i) \geq 0, \sum_i \pi_0(i) = 1 \)

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\( \{X_n, n \geq 0\} \) is defined so that:

\[ Pr[X_0 = i] = \pi_0(i), i \in \mathcal{X} \) (initial distribution)
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$\{X_n, n \geq 0\}$ is defined so that:

- $Pr[X_0 = i] = \pi_0(i), i \in \mathcal{X}$ (initial distribution)
- $Pr[X_{n+1} = j \mid X_0, \ldots, X_n = i] = P(i, j), i, j \in \mathcal{X}$. 
One Small (Time)step for a State

At each timestep \( t \) we are in some state \( X_t \in \mathcal{X} \).
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Where do we go next?

$$\Pr[X_{t+1} = j | X_t = i] = P_{i,j}$$
One Small (Time)step for a State

At each timestep $t$ we are in some state $X_t \in \mathcal{X}$. (random variable.)

Where do we go next?

$$
\Pr[X_{t+1} = j | X_t = i] = P_{i,j}
$$

Probability depends on the previous state, but is independent of how it got to the previous state. (It’s not independent of states before the previous state - but any dependence is captured in the previous state.)
At some point we might have a distribution for $X_t$ - say, it’s 1 w.p. 0.2, 2 w.p. 0.3, and 3 w.p. 0.5.
One Giant Leap with Conditional Probability

At some point we might have a distribution for $X_t$ - say, it’s 1 w.p. 0.2, 2 w.p. 0.3, and 3 w.p. 0.5. Distribution for $X_{t+1}$?
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$$Pr[X_{t+1} = 1] =$$
At some point we might have a distribution for $X_t$ - say, it’s 1 w.p. 0.2, 2 w.p. 0.3, and 3 w.p. 0.5. Distribution for $X_{t+1}$? Probability that it goes to 1?

$$\Pr[X_{t+1} = 1] = \sum_i \Pr[X_{t+1} = 1 | X_t = i] \Pr[X_t = i]$$
At some point we might have a distribution for $X_t$ - say, it’s 1 w.p. 0.2, 2 w.p. 0.3, and 3 w.p. 0.5. Distribution for $X_{t+1}$? Probability that it goes to 1?

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$$= 0.9 \times 0.2 + 0 \times 0.3 + 0.1 \times 0.5$$
At some point we might have a distribution for $X_t$ - say, it’s 1 w.p. 0.2, 2 w.p. 0.3, and 3 w.p. 0.5. Distribution for $X_{t+1}$? Probability that it goes to 1?

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$$= 0.9 \times 0.2 + 0 \times 0.3 + 0.1 \times 0.5 = 0.23$$

Rest of distribution for $X_{t+1}$ can be found similarly.
Very quick linear algebra intro:

Matrices: two-dimensional collection of numbers. A $n \times m$ matrix has $n$ rows, $m$ columns. Element at $i$th row, $j$th column denoted $A_{ij}$.

\[
\begin{bmatrix}
1 & 6 & 7 & 2 \\
6 & 5 & 6 & 3 \\
8 & 6 & 2 & 2 \\
2 & 5 & 3 & 8
\end{bmatrix}
\]

Vector: one-dimensional collection of numbers. We deal with row vectors - $n \times 1$ matrices.

\[
\begin{bmatrix}
5 & 9 & 3 & 0
\end{bmatrix}
\]
Very quick linear algebra intro:

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Matrix Multiplication

For $n \times m$ matrix $A$ and $m \times p$ matrix $B$:

$$(AB)_{ij} = \sum_k A_{ik}B_{kj}$$

\[
\begin{bmatrix}
5 & 9 & 3 & 0 \\
2 & 6 & 6 & 4 \\
1 & 6 & 7 & 2 \\
6 & 5 & 6 & 3 \\
8 & 6 & 2 & 2 \\
2 & 5 & 3 & 8 \\
3 & 7 & 7 & 5 \\
2 & 6 & 5 & 0 \\
6 & 9 & 3 & 0 \\
2 & 5 & 3 & 8 \\
3 & 7 & 7 & 5 \\
\end{bmatrix}
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Matrix Multiplication

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Or for vector \( x \):

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(xA)_i = \sum_k x_kA_{ki}
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\begin{bmatrix} 5 & 9 & 3 & 0 \end{bmatrix} \begin{bmatrix} 1 & 6 & 7 & 2 \\ 6 & 5 & 6 & 3 \\ 8 & 6 & 2 & 2 \\ 2 & 5 & 3 & 8 \end{bmatrix} = \begin{bmatrix} 1 \times 5 + 6 \times 9 + 8 \times 3 + 2 \times 0 \\ 6 \times 5 + 5 \times 9 + 6 \times 3 + 5 \times 0 \\ 7 \times 5 + 6 \times 9 + 2 \times 3 + 3 \times 0 \\ 2 \times 5 + 3 \times 9 + 2 \times 3 + 8 \times 0 \end{bmatrix}^T
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Matrix Multiplication

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$$\begin{bmatrix} 5 & 9 & 3 & 0 \end{bmatrix} \begin{bmatrix} 1 & 6 & 7 & 2 \\ 6 & 5 & 6 & 3 \\ 8 & 6 & 2 & 2 \\ 2 & 5 & 3 & 8 \end{bmatrix} = \begin{bmatrix} 1 \cdot 5 + 6 \cdot 9 + 8 \cdot 3 + 2 \cdot 0 \\ 6 \cdot 5 + 5 \cdot 9 + 6 \cdot 3 + 5 \cdot 0 \\ 7 \cdot 5 + 6 \cdot 9 + 2 \cdot 3 + 3 \cdot 0 \\ 2 \cdot 5 + 3 \cdot 9 + 2 \cdot 3 + 8 \cdot 0 \end{bmatrix}^T$$
Markov chains have a very nice translation to matrices! Transition probabilities form an *transition matrix* $P$ whose $i,j$th entry is $P_{i,j}$.

$$P = \begin{bmatrix} 0.9 & 0.1 & 0 \\ 0 & 0.4 & 0.6 \\ 0.1 & 0.4 & 0.5 \end{bmatrix}$$

Probabilities from a state sum to 1...
Markov chains have a very nice translation to matrices! Transition probabilities form an *transition matrix* $P$ whose $i,j$th entry is $P_{i,j}$.

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Matrix Markov

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Probabilities from a state sum to 1... rows sum to 1... *(right) stochastic matrix.*
Stepping with Multiplication

\[ P = \begin{bmatrix} 0.9 & 0.1 & 0 \\ 0 & 0.4 & 0.6 \\ 0.1 & 0.4 & 0.5 \end{bmatrix} \]

Distributions are vectors. Suppose that \( X_t \) is distributed with probability 0.2, 2 with probability 0.3, and 3 with probability 0.5. Write distribution as vector!

\[ t = \begin{bmatrix} 0.2 \\ 2 \\ 0.3 \\ 3 \\ 0.5 \end{bmatrix} \]

What's the product of \( t \) and \( P \)?

\[ 2640.9 + 0.3 + 0.5 + 2645 = 375 \]

\[ T = \begin{bmatrix} 0.23 \\ 0.34 \\ 0.43 \end{bmatrix} \]

This is the distribution of \( X_{t+1} \).
Distributions are vectors. Suppose that $X_t$ is distributed 1 w.p. 0.2, 2 w.p. 0.3, and 3 w.p. 0.5. Write distribution as vector!

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\[ \pi_t = \begin{bmatrix} 0.2 & 0.3 & 0.5 \end{bmatrix} \]
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What’s the product of \( \pi_t \) and \( P \)?

\[ \begin{bmatrix} 0.2 * 0.9 + 0.3 * 0 + 0.5 * 0.1 \\ 0.2 * 0.1 + 0.3 * 0.4 + 0.5 * 0.4 \\ 0.2 * 0 + 0.3 * 0.6 + 0.5 * 0.5 \end{bmatrix}^T \]
Stepping with Multiplication

\[ P = \begin{bmatrix} 0.9 & 0.1 & 0 \\ 0 & 0.4 & 0.6 \\ 0.1 & 0.4 & 0.5 \end{bmatrix} \]

Distributions are vectors. Suppose that \( X_t \) is distributed 1 w.p. 0.2, 2 w.p. 0.3, and 3 w.p. 0.5. Write distribution as vector!

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What’s the product of \( \pi_t \) and \( P \)?

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0.2 \times 0.1 + 0.3 \times 0.4 + 0.5 \times 0.4 \\
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Stepping with Multiplication

$P = \begin{bmatrix} 0.9 & 0.1 & 0 \\ 0 & 0.4 & 0.6 \\ 0.1 & 0.4 & 0.5 \end{bmatrix}$

Distributions are vectors. Suppose that $X_t$ is distributed 1 w.p. 0.2, 2 w.p. 0.3, and 3 w.p. 0.5. Write distribution as vector!

$\pi_t = \begin{bmatrix} 0.2 & 0.3 & 0.5 \end{bmatrix}$

What’s the product of $\pi_t$ and $P$?

$\begin{bmatrix} 0.2 \times 0.9 + 0.3 \times 0 + 0.5 \times 0.1 \\ 0.2 \times 0.1 + 0.3 \times 0.4 + 0.5 \times 0.4 \\ 0.2 \times 0 + 0.3 \times 0.6 + 0.5 \times 0.5 \end{bmatrix}^T = \begin{bmatrix} 0.23 & 0.34 & 0.43 \end{bmatrix}$

This is the distribution of $X_{t+1}$. 
Multiple Steps with Matrix Powers

One step: $\pi_t \rightarrow \pi_t P$

What if we take two steps? What's the distribution?

$t \rightarrow (tP)^2$

$n$ steps?

$tP^n$.

This will be very useful when we start talking about limiting distributions (next lecture).
Multiple Steps with Matrix Powers

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What if we take two steps? What’s the distribution?

$\pi_t \rightarrow (\pi_t P)P$
Multiple Steps with Matrix Powers

One step: \( \pi_t \rightarrow \pi_t P \)

What if we take two steps? What’s the distribution?
\( \pi_t \rightarrow (\pi_t P)P = \pi_t P^2 \)
One step: $\pi_t \rightarrow \pi_t P$

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$n$ steps? $\pi_t P^n$. 
One step: $\pi_t \rightarrow \pi_t P$

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$\pi_t \rightarrow (\pi_t P)P = \pi_t P^2$

$n$ steps? $\pi_t P^n$.

This will be very useful when we start talking about limiting distributions (next lecture).
An Example

California driving test: you get 3 retakes before you have to start the application process all over again. Suppose someone passes a driving test w.p. 0.6, unless it’s their final retake, in which case they’re more careful and pass w.p. 0.8.
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An Example

Initial distribution? $\pi_0 = [1 \ 0 \ 0 \ 0]$

Transition matrix?

$$T = \begin{bmatrix} 0 & 0.4 & 0 & 0.6 \\ 0 & 0 & 0.4 & 0.6 \\ 0.2 & 0 & 0 & 0.8 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$
Hitting Time
Motivation

How long does it take to get a driver’s license, in expectation?
How long does it take to get a driver’s license, in expectation?

Generally: given a Markov chain and an initial distribution, how many timesteps do we expect to take before reaching a particular state?
Let’s flip a coin with $Pr[H] = p$ until we get $H$. How many flips, on average?

Let $(S)$ be the average time until $E$, starting from $S$.

Then, $(S) = 1 + q(S) + p(0)$.

Hence, $p(S) = 1$; so that $(S) = 1/p$.

Note: Time until $E$ is $G(p)$.

We have rediscovered that the mean of $G(p)$ is $1/p$. 

15
A Simple Example

Let’s flip a coin with $Pr[H] = p$ until we get $H$. How many flips, on average?

Let $\beta(S)$ be the average time until $E$, starting from $S$. 

$q = 1 - p$

$p$

Let $\beta(S)$ be the average time until $E$, starting from $S$. 

$\beta(S) = 1 + q \cdot \beta(S) + p \cdot 0$

Hence, $p \cdot \beta(S) = 1$;

so that $\beta(S) = \frac{1}{p}$.

Note: Time until $E$ is $G(p)$. We have rediscovered that the mean of $G(p)$ is $\frac{1}{p}$. 

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Hence,

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![Diagram](image)

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$$\beta(S) = 1 + q\beta(S) + p0.$$ 

Hence,

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Let’s flip a coin with $Pr[H] = p$ until we get $H$. How many flips, on average?

Let $\beta(S) = \text{the average time until } E, \text{ starting from } S$. Then,

$$\beta(S) = 1 + q\beta(S) + p0.$$ 

Hence,

$$p\beta(S) = 1, \text{ so that } \beta(S) = 1/p.$$ 

Note: Time until $E$ is $G(p)$. We have rediscovered that the mean of $G(p)$ is $1/p$. 

\[ q = 1 - p \] 

\[ p \]

\[ S \]

\[ E \]

\[ X_0 \]
How Long to Get a Driver’s License?

Let $S$ denote expected time to get a driver’s license from $S$. 

$$(1) = 1 + 0.6 + 0.4$$

$$(2) = 1 + 0.6 + 0.8$$

$$(3) = 1 + 0.6 + 0.2$$

Solves to $(1) = 1.61$. 

```plaintext

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16
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How Long to Get a Driver’s License?

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\[
\begin{align*}
\beta(1) & = 1 + 0.6 \times 0 + 0.4 \times \beta(2) \\
\beta(2) & = 1 + 0.6 \times 0 + 0.4 \times \beta(3) \\
\beta(3) & = 1 + 0.8 \times 0 + 0.2 \times \beta(1)
\end{align*}
\]
Let $\beta(S)$ denote expected time to get a driver’s license from $S$.

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Solves to $\beta(1) \approx 1.61$. 

How Long to Get a Driver’s License?
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Driving test

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$$n = 1 + p(n + 1) + q(0); \quad 0 < n < 19$$

$$19 = 1 + p0 + q(0)$$

$$q = 1 - p$$

See Lecture Note 24 for algebra.
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\]

\[
\beta(19) = 1 + p\cdot 0 + q\beta(0)
\]

\[
\Rightarrow \beta(0) = \frac{p^{-20} - 1}{1-p}
\]
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\[
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\[ \Rightarrow \beta(0) = \frac{p^{-20} - 1}{1 - p} \approx 72. \]

See Lecture Note 24 for algebra.
Gig: Random names, random headlines