Random Variables: Expectation, Variance

1. Random Variables, Expectation: Brief Review
2. Independent Random Variables.
3. Variance
Random Variables: Definitions

Definition
A random variable, $X$, for a random experiment with sample space $\Omega$ is a variable that takes as value one of the random samples. NO!
Random Variables: Definitions

Definition
A random variable, $X$, for a random experiment with sample space $\Omega$ is a function $X : \Omega \rightarrow \mathbb{R}$.

Thus, $X(\cdot)$ assigns a real number $X(\omega)$ to each $\omega \in \Omega$.

Definitions
(a) For $a \in \mathbb{R}$, one defines the event
$$X^{-1}(a) := \{ \omega \in \Omega \mid X(\omega) = a \}.$$ 
(b) For $A \subset \mathbb{R}$, one defines the event
$$X^{-1}(A) := \{ \omega \in \Omega \mid X(\omega) \in A \}.$$ 
(c) The probability that $X = a$ is defined as
$$Pr[X = a] = Pr[X^{-1}(a)].$$ 
(d) The probability that $X \in A$ is defined as
$$Pr[X \in A] = Pr[X^{-1}(A)].$$ 
(e) The distribution of a random variable $X$, is
$$\{(a, Pr[X = a]) : a \in \mathcal{A}\},$$
where $\mathcal{A}$ is the range of $X$. That is, $\mathcal{A} = \{X(\omega), \omega \in \Omega\}$. 

An Example

Flip a fair coin three times.
\[\Omega = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}\].
\[X = \text{number of } H\text{'s}: \{3, 2, 2, 2, 1, 1, 1, 0\}\].

- **Range of** \(X\)? \(\{0, 1, 2, 3\}\). All the values \(X\) can take.
- **\(X^{-1}(2)\)?** \(X^{-1}(2) = \{HHT, HTH, THH\}\). All the outcomes \(\omega\) such that \(X(\omega) = 2\).
- **Is** \(X^{-1}(1)\) **an event?** YES. It’s a subset of the outcomes.
- **\(Pr[X]\)?** This doesn’t make any sense bro....
- **\(Pr[X = 2]\)?**

\[
Pr[X = 2] = Pr[X^{-1}(2)] = Pr[\{HHT, HTH, THH\}]
\]

\[
= Pr[\{HHT\}] + Pr[\{HTH\}] + Pr[\{THH\}] = \frac{3}{8}
\]
Random Variables: Definitions

Let $X, Y, Z$ be random variables on $\Omega$ and $g : \mathbb{R}^3 \to \mathbb{R}$ a function. Then $g(X, Y, Z)$ is the random variable that assigns the value $g(X(\omega), Y(\omega), Z(\omega))$ to $\omega$.
Thus, if $V = g(X, Y, Z)$, then $V(\omega) := g(X(\omega), Y(\omega), Z(\omega))$.

Examples:

- $X^k$
- $(X - a)^2$
- $a + bX + cX^2 + (Y - Z)^2$
- $(X - Y)^2$
- $X \cos(2\pi Y + Z)$. 
**Definition:** The expected value (or mean, or expectation) of a random variable $X$ is

$$E[X] = \sum_a a \times Pr[X = a].$$

**Theorem:**

$$E[X] = \sum_\omega X(\omega) \times Pr[\omega].$$
An Example

Flip a fair coin three times.
\( \Omega = \{ HHH, HHT, HTH, THH, HTT, THT, TTH, TTT \} \). \( X \) = number of \( H \)'s: \( \{3, 2, 2, 2, 1, 1, 1, 0\} \). Thus,

\[
\sum_{\omega} X(\omega)Pr[\omega] = 3 \frac{1}{8} + 2 \frac{1}{8} + 2 \frac{1}{8} + 2 \frac{1}{8} + 1 \frac{1}{8} + 1 \frac{1}{8} + 1 \frac{1}{8} + 0 \frac{1}{8}.
\]

Also,

\[
\sum_{a} a \times Pr[X = a] = 3 \frac{1}{8} + 2 \frac{3}{8} + 1 \frac{3}{8} + 0 \frac{1}{8}.
\]
Win or Lose.

Expected winnings for heads/tails games, with 3 flips?
Recall the definition of the random variable $X$:
$\{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\} \rightarrow \{3, 1, 1, -1, 1, -1, -1, -3\}$.

$$E[X] = 3\frac{1}{8} + 1\frac{3}{8} - 1\frac{3}{8} - 3\frac{1}{8} = 0.$$  

Can you ever win 0?

Apparently: Expected value is not a common value. It doesn’t have to be in the range of $X$.

The expected value of $X$ is not the value that you expect! It is the average value per experiment, if you perform the experiment many times. Let $X_1$ be your winnings the first time you play the game, $X_2$ are your winnings the second time you play the game, and so on. (Notice that $X_i$'s have the same distribution!) When $n \gg 1$:
$$\frac{X_1 + \cdots + X_n}{n} \rightarrow 0$$

The fact that this average converges to $E[X]$ is a theorem: the Law of Large Numbers. (See later.)
Law of Large Numbers

An Illustration: Rolling Dice
Indicators

Definition
Let $A$ be an event. The random variable $X$ defined by

$$X(\omega) = \begin{cases} 
1, & \text{if } \omega \in A \\
0, & \text{if } \omega \notin A 
\end{cases}$$

is called the indicator of the event $A$.

Note that $Pr[X = 1] = Pr[A]$ and $Pr[X = 0] = 1 - Pr[A]$.

Hence,

$$E[X] = 1 \times Pr[X = 1] + 0 \times Pr[X = 0] = Pr[A].$$

This random variable $X(\omega)$ is sometimes written as

$$1\{\omega \in A\} \text{ or } 1_A(\omega).$$

Thus, we will write $X = 1_A$. 

Linearity of Expectation

**Theorem:** Expectation is linear

\[ E[a_1 X_1 + \cdots + a_n X_n] = a_1 E[X_1] + \cdots + a_n E[X_n]. \]

**Proof:**

\[
E[a_1 X_1 + \cdots + a_n X_n] \\
= \sum_{\omega} (a_1 X_1 + \cdots + a_n X_n)(\omega)Pr[\omega] \\
= \sum_{\omega} (a_1 X_1(\omega) + \cdots + a_n X_n(\omega))Pr[\omega] \\
= a_1 \sum_{\omega} X_1(\omega)Pr[\omega] + \cdots + a_n \sum_{\omega} X_n(\omega)Pr[\omega] \\
= a_1 E[X_1] + \cdots + a_n E[X_n].
\]

Note: If we had defined \( Y = a_1 X_1 + \cdots + a_n X_n \) has had tried to compute \( E[Y] = \sum_y yPr[Y = y] \), we would have been in trouble!
Using Linearity - 1: Dots on dice

Roll a die $n$ times.

$X_m =$ number of dots on roll $m$.

$X = X_1 + \cdots + X_n =$ total number of dots in $n$ rolls.

\[
E[X] = E[X_1 + \cdots + X_n] \\
= E[X_1] + \cdots + E[X_n], \text{ by linearity} \\
= nE[X_1], \text{ because the } X_m \text{ have the same distribution}
\]

Now,

\[
E[X_1] = 1 \times \frac{1}{6} + \cdots + 6 \times \frac{1}{6} = \frac{6 \times 7}{2} \times \frac{1}{6} = \frac{7}{2}.
\]

Hence,

\[
E[X] = \frac{7n}{2}.
\]

Note: Computing $\sum_x xPr[X = x]$ directly is not easy!
Using Linearity - 2: Expected number of times a word appears.

Alex is typing a document randomly: Each letter has a probability of $\frac{1}{26}$ of being types. The document will be 100,000,000 letters long. What is the expected number of times that the word ”pizza” will appear?

Let $X$ be a random variable that counts the number of times the word ”pizza” appears. We want $E(X)$.

$$E(X) = \sum_{\omega} X(\omega) Pr[\omega].$$

Better approach: Let $X_i$ be the indicator variable that takes value 1 if ”pizza” starts on the $i$-th letter, and 0 otherwise. $i$ takes from 1 to $100,000 - 4 = 999,999,996$.

hpizzafgnpizzadjgbidgne....

$X_2 = 1, X_{10} = 1,...$
Using Linearity - 2: Expected number of times a word appears.

\[ E(X_i) = \left( \frac{1}{26} \right)^5 \]

Therefore,

\[ E(X) = E(\sum_i X_i) = \sum_i E(X_i) = 999,999,996 \left( \frac{1}{26} \right)^5 \approx 84 \]
Using Linearity - 3: The birthday paradox

Let $X$ be the random variable indicating the number of pairs of people, in a group of $k$ people, sharing the same birthday. What’s $E(X)$?

Let $X_{i,j}$ be the indicator random variable for the event that two people $i$ and $j$ have the same birthday. $X = \sum_{i,j} X_{i,j}$.

$$E[X] = E[\sum_{i,j} X_{i,j}]$$
$$= \sum_{i,j} E[X_{i,j}]$$
$$= \sum_{i,j} Pr[X_{i,j}]$$
$$= \sum_{i,j} \frac{1}{365} = \binom{k}{2} \frac{1}{365} = k(k-1) \frac{1}{2 \cdot 365}$$

For a group of 28 it’s about 1. For 100 it’s 13.5. For 280 it’s 107.
Calculating $E[g(X)]$

Let $Y = g(X)$. Assume that we know the distribution of $X$.

We want to calculate $E[Y]$.

**Method 1:** We calculate the distribution of $Y$:

$$Pr[Y = y] = Pr[X \in g^{-1}(y)]$$
where $g^{-1}(x) = \{x \in \mathbb{R} : g(x) = y\}$.

This is typically rather tedious!

**Method 2:** We use the following result.

**Theorem:**

$$E[g(X)] = \sum_{x \in \mathcal{A}(X)} g(x) Pr[X = x].$$

**Proof:**

$$E[g(X)] = \sum_{\omega} g(X(\omega)) Pr[\omega] = \sum_{x} \sum_{\omega \in X^{-1}(x)} g(X(\omega)) Pr[\omega]$$

$$= \sum_{x} \sum_{\omega \in X^{-1}(x)} g(x) Pr[\omega] = \sum_{x} g(x) \sum_{\omega \in X^{-1}(x)} Pr[\omega]$$

$$= \sum_{x} g(x) Pr[X = x].$$
An Example

Let $X$ be uniform in $\{-2, -1, 0, 1, 2, 3\}$. Let also $g(X) = X^2$. Then (method 2)

$$E[g(X)] = \sum_{x=-2}^{3} x^2 \frac{1}{6}$$

$$= \left\{ 4 + 1 + 0 + 1 + 4 + 9 \right\} \frac{1}{6} = \frac{19}{6}.$$ 

Method 1 - We find the distribution of $Y = X^2$:

$$Y = \begin{cases} 
4, & \text{w.p. } \frac{2}{6} \\
1, & \text{w.p. } \frac{1}{6} \\
0, & \text{w.p. } \frac{1}{6} \\
9, & \text{w.p. } \frac{1}{6}.
\end{cases}$$

Thus,

$$E[Y] = 4 \frac{2}{6} + 1 \frac{2}{6} + 0 \frac{1}{6} + 9 \frac{1}{6} = \frac{19}{6}.$$
Calculating $E[g(X, Y, Z)]$

We have seen that $E[g(X)] = \sum_x g(x) Pr[X = x]$.

Using a similar derivation, one can show that

$$E[g(X, Y, Z)] = \sum_{x,y,z} g(x, y, z) Pr[X = x, Y = y, Z = z] .$$

**An Example.** Let $X, Y$ be as shown below:

$$E[\cos(2\pi X + \pi Y)] = 0.1 \cos(0) + 0.4 \cos(2\pi) + 0.2 \cos(\pi) + 0.3 \cos(3\pi) = 0.1 \times 1 + 0.4 \times 1 + 0.2 \times (-1) + 0.3 \times (-1) = 0.$$
The expected value has a *center of mass* interpretation:

\[
\sum_n p_n (a_n - \mu) = 0
\]

\[
\Leftrightarrow \mu = \sum_n a_n p_n = E[X]
\]
If you only know the distribution of $X$, it seems that $E[X]$ is a ‘good guess’ for $X$.

The following result makes that idea precise.

**Theorem**
The value of $a$ that minimizes $E[(X - a)^2]$ is $a = E[X]$.

Unfortunately, we won’t talk about this in this class...
Definition: Independence

The random variables $X$ and $Y$ are independent if and only if

$$Pr[Y = b | X = a] = Pr[Y = b],$$
for all $a$ and $b$.

Fact:

$X$, $Y$ are independent if and only if

$$Pr[X = a, Y = b] = Pr[X = a]Pr[Y = b],$$
for all $a$ and $b$.

Obvious.
Independence: Examples

Example 1
Roll two die. $X =$ number of dots on the first one, $Y =$ number of dots on the other one. $X$, $Y$ are independent.

Indeed: $Pr[X = a, Y = b] = \frac{1}{36}, Pr[X = a] = Pr[Y = b] = \frac{1}{6}$.

Example 2
Roll two die. $X =$ total number of dots, $Y =$ number of dots on die 1 minus number on die 2. $X$ and $Y$ are not independent.

Indeed: $Pr[X = 12, Y = 1] = 0 \neq Pr[X = 12]Pr[Y = 1] > 0$. 
Theorem Functions of independent RVs are independent
Let $X, Y$ be independent RV. Then

$$f(X) \text{ and } g(Y) \text{ are independent, for all } f(\cdot), g(\cdot).$$
**Theorem**

Let $X, Y$ be independent RVs. Then

$$E[XY] = E[X]E[Y].$$

**Proof:**

Recall that $E[g(X, Y)] = \sum_{x, y} g(x, y) Pr[X = x, Y = y]$. Hence,

$$E[XY] = \sum_{x, y} xyPr[X = x, Y = y] = \sum_{x, y} xyPr[X = x]Pr[Y = y], \text{ by ind.}$$

$$= \sum_x \left[ \sum_y xyPr[X = x]Pr[Y = y] \right] = \sum_x \left[ xPr[X = x] \sum_y yPr[Y = y] \right]$$

$$= \sum_x \left[ xPr[X = x]E[Y] \right] = E[X]E[Y].$$

$\square$
Examples


Wait. Isn’t $X$ independent with itself? No. If I tell you the value of $X$, then you know the value of $X$.

Then

$$E[(X + 2Y + 3Z)^2] = E[X^2 + 4Y^2 + 9Z^2 + 4XY + 12YZ + 6XZ]$$

$$= 1 + 4 + 9 + 4 \times 0 + 12 \times 0 + 6 \times 0$$

$$= 14.$$ 

(2) Let $X$, $Y$ be independent and take values from \{1,2,...,n\} uniformly at random. Then


$$= \frac{1 + 3n + 2n^2}{3} - \frac{(n+1)^2}{2}.$$
Mutually Independent Random Variables

Definition
$X, Y, Z$ are mutually independent if

$$Pr[X = x, Y = y, Z = z] = Pr[X = x]Pr[Y = y]Pr[Z = z],$$
for all $x, y, z$.

Theorem
The events $A, B, C, \ldots$ are pairwise (resp. mutually) independent iff the random variables $1_A, 1_B, 1_C, \ldots$ are pairwise (resp. mutually) independent.

Proof:
$$Pr[1_A = 1, 1_B = 1, 1_C = 1] = Pr[A \cap B \cap C], \ldots$$
If $X$, $Y$, $Z$ are pairwise independent, but not mutually independent, it may be that

$$f(X)$$ and $$g(Y, Z)$$ are not independent.

**Example:** Flip two fair coins,
$X = 1\{\text{coin 1 is } H\}$, $Y = 1\{\text{coin 2 is } H\}$, $Z = X \oplus Y$. Then, $X$, $Y$, $Z$ are pairwise independent. Let $g(Y, Z) = Y \oplus Z$. Then $g(Y, Z) = X$ is not independent of $X$. 
Functions of mutually independent RVs

One has the following result:

**Theorem**
Functions of disjoint collections of mutually independent random variables are mutually independent.

**Example:**
Let \( \{X_n, n \geq 1\} \) be mutually independent. Then,
\[
Y_1 := X_1 X_2 (X_3 + X_4)^2, \quad Y_2 := \max\{X_5, X_6\} - \min\{X_7, X_8\}, \quad Y_3 := X_9 \cos(X_{10} + X_{11})
\]
are mutually independent.

**Proof:**
Let \( B_1 := \{(x_1, x_2, x_3, x_4) \mid x_1 x_2 (x_3 + x_4)^2 \in A_1\} \). Similarly for \( B_2, B_3 \).

Then
\[
Pr[Y_1 \in A_1, Y_2 \in A_2, Y_3 \in A_3]
= Pr[(X_1, \ldots, X_4) \in B_1, (X_5, \ldots, X_8) \in B_2, (X_9, \ldots, X_{11}) \in B_3]
= Pr[(X_1, \ldots, X_4) \in B_1]Pr[(X_5, \ldots, X_8) \in B_2]Pr[(X_9, \ldots, X_{11}) \in B_3]
= Pr[Y_1 \in A_1]Pr[Y_2 \in A_2]Pr[Y_3 \in A_3]
\]

Theorem

Operations on disjoint collections of mutually independent events produce mutually independent events.

For instance, if $A, B, C, D, E$ are mutually independent, then $A \Delta B, C \setminus D, \bar{E}$ are mutually independent.
Product of mutually independent RVs

**Theorem**
Let $X_1, \ldots, X_n$ be mutually independent RVs. Then,

$$E[X_1 X_2 \cdots X_n] = E[X_1] E[X_2] \cdots E[X_n].$$

**Proof:**
Assume that the result is true for $n$. (It is true for $n = 2$.) Then, with $Y = X_1 \cdots X_n$, one has

$$E[X_1 \cdots X_n X_{n+1}] = E[Y X_{n+1}],$$

$$= E[Y] E[X_{n+1}],$$

because $Y, X_{n+1}$ are independent

$$= E[X_1] \cdots E[X_n] E[X_{n+1}].$$
Flip a coin: If H you make a dollar. If T you lose a dollar. Let $X$ be the RV indicating how much money you make. $E(X) = 0$.

Flip a coin: If H you make a million dollars. If T you lose a million dollars. Let $Y$ be the RV indicating how much money you make. $E(Y) = 0$.

Any other measures?? What else that’s informative can we say?
The variance measures the deviation from the mean value.

**Definition:** The variance of $X$ is

$$\sigma^2(X) := \text{var}[X] = E[(X - E[X])^2].$$

$\sigma(X)$ is called the standard deviation of $X$. 
Variance and Standard Deviation

Fact:


Indeed:

$$\text{var}(X) = E[(X - E[X])^2]$$  

$$= E[X^2 - 2XE[X] + E[X]^2]$$  

$$= E[X^2] - E[2XE[X]] + E[E[X]^2] \text{ by linearity}$$  

$$= E[X^2] - 2E[X]E[X] + E[X]^2,$$  

A simple example

This example illustrates the term ‘standard deviation.’

Consider the random variable $X$ such that

$$X = \begin{cases} 
\mu - \sigma, & \text{w.p. } 1/2 \\
\mu + \sigma, & \text{w.p. } 1/2.
\end{cases}$$

Then, $E[X] = \mu$ and $(X - E[X])^2 = \sigma^2$. Hence,

$$\text{var}(X) = \sigma^2 \text{ and } \sigma(X) = \sigma.$$
Example

Consider $X$ with

$$X = \begin{cases} 
  -1, & \text{w. p. 0.99} \\
  99, & \text{w. p. 0.01}.
\end{cases}$$

Then

$$E[X] = -1 \times 0.99 + 99 \times 0.01 = 0.$$  
$$E[X^2] = 1 \times 0.99 + (99)^2 \times 0.01 \approx 100.$$  
$$Var(X) \approx 100 \implies \sigma(X) \approx 10.$$
Properties of variance.

1. $\text{Var}(cX) = c^2 \text{Var}(X)$, where $c$ is a constant. 
   Scales by $c^2$.

2. $\text{Var}(X + c) = \text{Var}(X)$, where $c$ is a constant. 
   Shifts center.

Proof:

\[
\text{Var}(cX) = E((cX)^2) - (E(cX))^2 \\
= c^2 E(X^2) - c^2 (E(X))^2 = c^2 (E(X^2) - E(X)^2) \\
= c^2 \text{Var}(X)
\]

\[
\text{Var}(X + c) = E((X + c - E(X + c))^2) \\
= E((X + c - E(X) - c)^2) \\
= E((X - E(X))^2) = \text{Var}(X)
\]

\[
\square
\]
Variance of sum of two independent random variables

**Theorem:**
If $X$ and $Y$ are independent, then

$$\text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y).$$

**Proof:**
Since shifting the random variables does not change their variance, let us subtract their means.

That is, we assume that $E(X) = 0$ and $E(Y) = 0$.

Then, by independence,

$$E(XY) = E(X)E(Y) = 0.$$ 

Hence,

$$\text{var}(X + Y) = E((X + Y)^2) = E(X^2 + 2XY + Y^2)$$
$$= E(X^2) + 2E(XY) + E(Y^2) = E(X^2) + E(Y^2)$$
$$= E(X^2) - (E(X))^2 + E(Y^2) - (E(Y))^2 = \text{var}(X) + \text{var}(Y).$$
Variance of sum of independent random variables

**Theorem:**
If $X, Y, Z, \ldots$ are pairwise independent, then

$$\text{var}(X + Y + Z + \cdots) = \text{var}(X) + \text{var}(Y) + \text{var}(Z) + \cdots.$$ 

**Proof:**
Since shifting the random variables does not change their variance, let us subtract their means.

That is, we assume that $E[X] = E[Y] = \cdots = 0$.

Then, by independence,

$$E[XY] = E[X]E[Y] = 0. \text{ Also, } E[XZ] = E[YZ] = \cdots = 0.$$ 

Hence,

$$\text{var}(X + Y + Z + \cdots) = E((X + Y + Z + \cdots)^2)$$ 
$$= E(X^2 + Y^2 + Z^2 + \cdots + 2XY + 2XZ + 2YZ + \cdots)$$ 
$$= E(X^2) + E(Y^2) + E(Z^2) + \cdots + 0 + \cdots + 0$$ 
$$= \text{var}(X) + \text{var}(Y) + \text{var}(Z) + \cdots.$$
Today’s gig: Lies!

Gigs so far:

1. How to tell random from human.
2. Monty Hall.
4. St. Petersburg paradox

Today: Simpson’s paradox.

How come this show is still around?

Wait... Wrong Simpson.
The paradox

In 1314 English women were surveyed in 1972-1974 and again after 20 years about smoking:

<table>
<thead>
<tr>
<th>Smoker</th>
<th>Dead</th>
<th>Alive</th>
<th>Total</th>
<th>% Dead</th>
</tr>
</thead>
<tbody>
<tr>
<td>Yes</td>
<td>139</td>
<td>443</td>
<td>582</td>
<td>24</td>
</tr>
<tr>
<td>No</td>
<td>230</td>
<td>502</td>
<td>732</td>
<td>31</td>
</tr>
<tr>
<td>Total</td>
<td>369</td>
<td>945</td>
<td>1314</td>
<td>28</td>
</tr>
</tbody>
</table>

Not smoking kills!
The paradox

A closer look:

<table>
<thead>
<tr>
<th>Age group</th>
<th>18–24</th>
<th>25–34</th>
<th>35–44</th>
<th>45–54</th>
<th>55–54</th>
</tr>
</thead>
<tbody>
<tr>
<td>Smoker</td>
<td>Y</td>
<td>N</td>
<td>Y</td>
<td>N</td>
<td>Y</td>
</tr>
<tr>
<td>Dead</td>
<td>2</td>
<td>1</td>
<td>3</td>
<td>5</td>
<td>11</td>
</tr>
<tr>
<td>Alive</td>
<td>53</td>
<td>61</td>
<td>121</td>
<td>152</td>
<td>95</td>
</tr>
<tr>
<td>Ratio</td>
<td>2.3</td>
<td>0.75</td>
<td>2.4</td>
<td>1.44</td>
<td>1.61</td>
</tr>
</tbody>
</table>

In each separate category, the percentage of fatalities among smokers is higher, and yet the overall percentage of fatalities among smokers is lower!
A random variable $X$ is a function $X : \Omega \rightarrow \mathbb{R}$.

$Pr[X = a] := Pr[X^{-1}(a)] = Pr[\{\omega \mid X(\omega) = a\}]$.

$Pr[X \in A] := Pr[X^{-1}(A)]$.

The distribution of $X$ is the list of possible values and their probability:

$\{(a, Pr[X = a]), a \in \mathcal{A}\}$.

$g(X, Y, Z)$ assigns the value .... .

$E[X] := \sum_a aPr[X = a]$.

Expectation is Linear.

Independent Random Variables.

Variance.