Bayes’ Rule, Mutual Independence, Collisions and Collecting
Alex Psomas: Lecture 15.

Bayes’ Rule, Mutual Independence, Collisions and Collecting

1. Conditional Probability
2. Independence
3. Bayes’ Rule
4. Balls and Bins
5. Coupons
Conditional Probability: Review

Recall:

\[ P[A \mid B] = \frac{P[A \cap B]}{P[B]} \]

Hence, \[ P[A \cap B] = P[B] \cdot P[A \mid B] = P[A] \cdot P[B \mid A] \].

- A and B are positively correlated if \( P[A \mid B] > P[A] \), i.e., if \( P[A \cap B] > P[A] \cdot P[B] \).
- A and B are negatively correlated if \( P[A \mid B] < P[A] \), i.e., if \( P[A \cap B] < P[A] \cdot P[B] \).
- A and B are independent if \( P[A \mid B] = P[A] \), i.e., if \( P[A \cap B] = P[A] \cdot P[B] \).

Note:

- If \( B \subset A \) and \( P[A] \neq 1 \), \( P[B] \neq 0 \), \( A \) and \( B \) are positively correlated.

\( (P[A \mid B] = 1 > P[A]) \)

Note:

- If \( A \cap B \neq \emptyset \), \( P[A] \neq 0 \), \( P[B] \neq 0 \), \( A \) and \( B \) are negatively correlated.

\( (P[A \mid B] = 0 < P[A]) \)
Conditional Probability: Review

Recall:

- \( \Pr[A|B] = \frac{Pr[A \cap B]}{Pr[B]} \).

- \( A \) and \( B \) are positively correlated if \( \Pr[A|B] > \Pr[A] \), i.e., if \( \Pr[A \cap B] > \Pr[A] \Pr[B] \).

- \( A \) and \( B \) are negatively correlated if \( \Pr[A|B] < \Pr[A] \), i.e., if \( \Pr[A \cap B] < \Pr[A] \Pr[B] \).

- \( A \) and \( B \) are independent if \( \Pr[A|B] = \Pr[A] \), i.e., if \( \Pr[A \cap B] = \Pr[A] \Pr[B] \).

- Note: \( B \subset A \), and \( \Pr[A] \neq 1 \), \( \Pr[B] \neq 0 \), \( \Rightarrow A \) and \( B \) are positively correlated (\( \Pr[A|B] = 1 > \Pr[A] \)).

- Note: \( A \cap B = \emptyset \), \( \Pr[A] \neq 0 \), \( \Pr[B] \neq 0 \), \( \Rightarrow A \) and \( B \) are negatively correlated (\( \Pr[A|B] = 0 < \Pr[A] \)).
Conditional Probability: Review

Recall:

- \( \Pr[A|B] = \frac{\Pr[A \cap B]}{\Pr[B]} \).

- Hence, \( \Pr[A \cap B] = \Pr[B]\Pr[A|B] = \Pr[A]\Pr[B|A] \).
Conditional Probability: Review

Recall:

- \( \Pr[A|B] = \frac{\Pr[A \cap B]}{\Pr[B]} \).
- Hence, \( \Pr[A \cap B] = \Pr[B] \Pr[A|B] = \Pr[A] \Pr[B|A] \).
- A and B are \textit{positively correlated} if \( \Pr[A|B] > \Pr[A] \),
Conditional Probability: Review

Recall:

- \( Pr[A|B] = \frac{Pr[A \cap B]}{Pr[B]} \).
- \( A \) and \( B \) are positively correlated if \( Pr[A|B] > Pr[A] \), i.e., if \( Pr[A \cap B] > Pr[A]Pr[B] \).
Conditional Probability: Review

Recall:

- $P(A|B) = \frac{P(A \cap B)}{P(B)}$.

- Hence, $P(A \cap B) = P(B)P(A|B) = P(A)P(B|A)$.

- $A$ and $B$ are positively correlated if $P(A|B) > P(A)$, i.e., if $P(A \cap B) > P(A)P(B)$.

- $A$ and $B$ are negatively correlated if $P(A|B) < P(A)$,
Conditional Probability: Review

Recall:

- $Pr[A|B] = \frac{Pr[A \cap B]}{Pr[B]}$.


- $A$ and $B$ are positively correlated if $Pr[A|B] > Pr[A]$, i.e., if $Pr[A \cap B] > Pr[A]Pr[B]$.

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- Note: $B \subset A$, and $\Pr[A] \neq 1$, $\Pr[B] \neq 0$, $\implies A$ and $B$ are...
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- $Pr[A|B] = \frac{Pr[A \cap B]}{Pr[B]}$.


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- Note: $B \subset A$, and $Pr[A] \neq 1$, $Pr[B] \neq 0$, $\Rightarrow A$ and $B$ are positively correlated.
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Recall:

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- Note: \( B \subset A \), and \( Pr[A] \neq 1, Pr[B] \neq 0 \), \( \Rightarrow \) \( A \) and \( B \) are positively correlated. (\( Pr[A|B] = 1 > Pr[A] \))
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- Note: $B \subset A$, and $Pr[A] \neq 1$, $Pr[B] \neq 0$, ⇒ $A$ and $B$ are positively correlated. ($Pr[A|B] = 1 > Pr[A]$)
- Note: $A \cap B = \emptyset$, $Pr[A]$, $Pr[B] \neq 0$, ⇒ $A$ and $B$ are
Conditional Probability: Review

Recall:

1. \( \Pr[A|B] = \frac{\Pr[A \cap B]}{\Pr[B]} \).
2. Hence, \( \Pr[A \cap B] = \Pr[B] \Pr[A|B] = \Pr[A] \Pr[B|A] \).
3. \( A \) and \( B \) are positively correlated if \( \Pr[A|B] > \Pr[A] \),
   i.e., if \( \Pr[A \cap B] > \Pr[A] \Pr[B] \).
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6. Note: \( B \subset A \), and \( \Pr[A] \neq 1, \Pr[B] \neq 0 \), \( \Rightarrow \) \( A \) and \( B \) are positively correlated. (\( \Pr[A|B] = 1 > \Pr[A] \))
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- Note: \( A \cap B = \emptyset \), \( Pr[A], Pr[B] \neq 0 \), \( \Rightarrow \) \( A \) and \( B \) are negatively correlated. \( (Pr[A|B] = 0 < Pr[A]) \)
Monty Hall

3 closed doors. Behind one of the doors there is a prize (car). The others have goats.

You pick a door. Say door number 1. I open door 2 or door 3. One of the two that I know doesn’t have the prize. Say it was door 2. I ask: Would you like to change your door to number 3?

Question: What should you do in order to maximize the probability of winning?
Monty Hall

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Monty Hall

Change!!!!

What is the probability that the prize is in door 3?

How does that make any sense???

Say the original door where the prize is random.

So each door has probability \( \frac{1}{3} \).

You pick door 1.

What's the probability that it's in either 2 or 3?

The door I opened wasn't random! I knew it didn't have a prize!!

Therefore, switching, is like getting to pick two doors at the beginning!
Monty Hall

Change!!!!
Monty Hall

Change!!!!

What is the probability that the prize is in door 3? \(\frac{2}{3}\)!
Change!!!!

What is the probability that the prize is in door 3? \( \frac{2}{3} \)

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Monty Hall

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Say the original door where the prize is random. So each door has probability $\frac{1}{3}$. 

Monty Hall

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The door I opened wasn’t random! I knew it didn’t have a prize!! Therefore, switching, is like getting to pick two doors at the beginning!
...and my yard has so much grass, and I'll teach you tricks, and...
I throw 5 (indistinguishable) balls in two bins.
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1. Approach 1: There are 6 outcomes: (5, 0), (4, 1), (3, 2), (2, 3), (1, 4), (0, 5).
Balls in bins

I throw 5 (indistinguishable) balls in two bins. What is the probability that the first bin is empty?

1. Approach 1: There are 6 outcomes: (5,0), (4,1), (3,2), (2,3), (1,4), (0,5). Probability that the first bin is empty is $\frac{1}{6}$.
I throw 5 (indistinguishable) balls in two bins. What is the probability that the first bin is empty?

1. Approach 1: There are 6 outcomes: (5, 0), (4, 1), (3, 2), (2, 3), (1, 4), (0, 5). Probability that the first bin is empty is $\frac{1}{6}$.

2. Approach 2: I pretend I can tell the balls apart. Probability that the first bin is empty is $\frac{1}{2}$.

The fact that I can tell them apart shouldn't change the probability. Well... I guess probability is wrong... Or...... Could one of the approaches be wrong???

Approach 1 is WRONG! Why did we divide by $|\Omega|$???
Balls in bins

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2. Approach 2: I pretend I can tell the balls apart. There are $2^5$ outcomes: $(1, 1, 1, 1, 1)$,
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1. Approach 1: There are 6 outcomes: (5, 0), (4, 1), (3, 2), (2, 3), (1, 4), (0, 5). Probability that the first bin is empty is \(\frac{1}{6}\).

2. Approach 2: I pretend I can tell the balls apart. There are \(2^5\) outcomes: (1, 1, 1, 1, 1), (1, 1, 1, 1, 2), \ldots (2, 2, 2, 2, 2). (\(x, 1, x, x, x\)) means that the second ball I threw landed in the first bin. Probability that the first bin is empty is \(\frac{1}{25}\).
Balls in bins

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Probability that the first bin is empty is $\frac{1}{2^5}$. The fact that I can tell them apart shouldn’t change the probability.

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Approach 1 is WRONG! Why did we divide by $|\Omega|$???
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Why???????? Nooooooooooooooo
Balls in bins

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Approach 1 is WRONG! Why did we divide by \(|\Omega|\)???

Why???????? Noooooooooooxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxx
Conditional Probability: Pictures

Left: A and B are independent. \( \Pr[B] = b_1; \Pr[B|A] = b_2 \).

Middle: A and B are positively correlated. \( \Pr[B|A] > \Pr[B|\bar{A}] = b_2 \). Note: \( \Pr[B] \in (b_2, b_1) \).

Right: A and B are negatively correlated. \( \Pr[B|A] < \Pr[B|\bar{A}] = b_2 \). Note: \( \Pr[B] \in (b_1, b_2) \).
Conditional Probability: Pictures

Illustrations: Pick a point uniformly in the unit square

Left: $A$ and $B$ are independent. $\Pr[B] = \frac{1}{2}$; $\Pr[B | A] = \frac{1}{2}$.

Middle: $A$ and $B$ are positively correlated. $\Pr[B | A] = \frac{1}{2}$ > $\Pr[B | \bar{A}] = \frac{1}{2}$. Note: $\Pr[B] \in (\frac{1}{2}, \frac{3}{4})$.

Right: $A$ and $B$ are negatively correlated. $\Pr[B | A] = \frac{1}{2}$ < $\Pr[B | \bar{A}] = \frac{1}{2}$. Note: $\Pr[B] \in (\frac{1}{4}, \frac{1}{2})$. 
Illustrations: Pick a point uniformly in the unit square

- **Left:** $A$ and $B$ are independent.
  \[
  \Pr(B) = b; \quad \Pr(B \mid A) = b.
  \]
  Note: $\Pr(B \in (b_2, b_1))$.

- **Middle:** $A$ and $B$ are positively correlated.
  \[
  \Pr(B \mid A) = b_1 > \Pr(B \mid \bar{A}) = b_2.
  \]

- **Right:** $A$ and $B$ are negatively correlated.
  \[
  \Pr(B \mid A) = b_1 < \Pr(B \mid \bar{A}) = b_2.
  \]
  Note: $\Pr(B \in (b_1, b_2))$. 

▶
Conditional Probability: Pictures

Illustrations: Pick a point uniformly in the unit square

- **Left:** $A$ and $B$ are independent.

- **Middle:** $A$ and $B$ are positively correlated.

- **Right:** $A$ and $B$ are negatively correlated.
Conditional Probability: Pictures

Illustrations: Pick a point uniformly in the unit square

▶ Left: $A$ and $B$ are independent. $Pr[B] =$

▶ Middle: $A$ and $B$ are positively correlated. $Pr[B | A] > Pr[B | \neg A]$.
Note: $Pr[B] \in (b_2, b_1)$.

▶ Right: $A$ and $B$ are negatively correlated. $Pr[B | A] < Pr[B | \neg A]$.
Note: $Pr[B] \in (b_1, b_2)$. 
Conditional Probability: Pictures

Illustrations: Pick a point uniformly in the unit square

- Left: $A$ and $B$ are independent. $Pr[B] = b$;

- Middle: $A$ and $B$ are positively correlated. $Pr[B \mid A] > Pr[B \mid \overline{A}]$.

- Right: $A$ and $B$ are negatively correlated. $Pr[B \mid A] < Pr[B \mid \overline{A}]$. Note: $Pr[B] \in (b_2, b_1)$.
Conditional Probability: Pictures

Illustrations: Pick a point uniformly in the unit square

Left: $A$ and $B$ are independent. $Pr[B] = b; Pr[B|A] =$

Middle:$A$ and $B$ are positively correlated. $Pr[B|A] > Pr[B|\bar{A}] = b_2$. Note: $Pr[B] \in (b_2, b_1)$.

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Conditional Probability: Pictures

Illustrations: Pick a point uniformly in the unit square

- **Left:** $A$ and $B$ are independent. $\Pr[B] = b; \Pr[B|A] = b$.
Conditional Probability: Pictures

Illustrations: Pick a point uniformly in the unit square

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Conditional Probability: Pictures

Illustrations: Pick a point uniformly in the unit square

- **Left**: $A$ and $B$ are independent. $Pr[B] = b; Pr[B|A] = b$.
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Conditional Probability: Pictures

Illustrations: Pick a point uniformly in the unit square

- **Left:** $A$ and $B$ are independent. $\Pr[B] = b; \Pr[B|A] = b$.  
- **Middle:** $A$ and $B$ are positively correlated. $\Pr[B|A] =$ 
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Conditional Probability: Pictures

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- **Right:** $A$ and $B$ are negatively correlated. $Pr[B|A] = b_1 < Pr[B|\bar{A}] = \cdots$
Illustrations: Pick a point uniformly in the unit square

- **Left:** $A$ and $B$ are independent. $Pr[B] = b; Pr[B|A] = b$.
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- **Right:** $A$ and $B$ are negatively correlated. $Pr[B|A] = b_1 < Pr[B|\bar{A}] = b_2$. 

Note: $Pr[B] \in (b_2, b_1)$.
Illustrations: Pick a point uniformly in the unit square

- **Left:** $A$ and $B$ are independent. $Pr[B] = b; Pr[B|A] = b$.
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Conditional Probability: Pictures

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- **Left:** $A$ and $B$ are independent. $Pr[B] = b; Pr[B|A] = b$.
- **Middle:** $A$ and $B$ are positively correlated.
  $Pr[B|A] = b_1 > Pr[B|\bar{A}] = b_2$. Note: $Pr[B] \in (b_2, b_1)$.
- **Right:** $A$ and $B$ are
Conditional Probability: Pictures

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Conditional Probability: Pictures

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  $Pr[B|A] = b_1 < Pr[B|\bar{A}] = b_2$. 
Conditional Probability: Pictures

Illustrations: Pick a point uniformly in the unit square

- **Left:** $A$ and $B$ are independent. $Pr[B] = b; Pr[B|A] = b$.

- **Middle:** $A$ and $B$ are positively correlated.
  
  $Pr[B|A] = b_1 > Pr[B|\bar{A}] = b_2$. Note: $Pr[B] \in (b_2, b_1)$.

- **Right:** $A$ and $B$ are negatively correlated.
  
  $Pr[B|A] = b_1 < Pr[B|\bar{A}] = b_2$. Note: $Pr[B] \in (b_1, b_2)$. 
Bayes and Biased Coin

Pick a point uniformly at random in the unit square. Then

\[ \Pr[A] = 0.5; \quad \Pr[\bar{A}] = 0.5 \]

\[ \Pr[B|A] = 0.5; \quad \Pr[B|\bar{A}] = 0.6 \]

\[ \Pr[A \cap B] = 0.5 \times 0.5 = \Pr[B] = 0.5 \times 0.5 + 0.5 \times 0.6 \approx 0.46 = \text{fraction of } B \text{ that is inside } A \]
Bayes and Biased Coin

\[ \Pr[A] = 0.5; \quad \Pr[\neg A] = 0.5 \]

\[ \Pr[B | A] = 0.5; \quad \Pr[B | \neg A] = 0.6 \]

\[ \Pr[A \cap B] = \Pr[A] \times \Pr[B | A] + \Pr[\neg A] \times \Pr[B | \neg A] \]

\[ \approx 0.46 \]

fraction of $B$ that is inside $A$
Bayes and Biased Coin

Pick a point uniformly at random in the unit square. Then
Bayes and Biased Coin

Pick a point uniformly at random in the unit square. Then

\[ Pr[A] = \]

\[ Pr[A] = Pr[B] \times \frac{Pr[A]}{Pr[A] + Pr[\bar{A}]} + Pr[\bar{A}] \times \frac{Pr[A]}{Pr[A] + Pr[\bar{A}]} \approx 0.46 \]

fraction of \( B \) that is inside \( A \)
Pick a point uniformly at random in the unit square. Then

$$Pr[A] = 0.5;$$
Bayes and Biased Coin

Pick a point uniformly at random in the unit square. Then

\[ P(A) = 0.5; \ P(\bar{A}) = \]

\[ P(A) = 0.5 \times 0.5 + 0.5 \times 0.6 = 0.56 \]

\[ P(B | A) = \]

\[ P(B | \bar{A}) = \]

\[ P(A \cap B) = P(A) P(B | A) + P(\bar{A}) P(B | \bar{A}) = 0.56 = P(A) \]

\[ \approx 0.46 \]

\[ \text{fraction of } B \text{ that is inside } A \]
Bayes and Biased Coin

Pick a point uniformly at random in the unit square. Then

\[ Pr[A] = 0.5; \quad Pr[\overline{A}] = 0.5 \]
Bayes and Biased Coin

Pick a point uniformly at random in the unit square. Then

\[ Pr[A] = 0.5; \ Pr[\overline{A}] = 0.5 \]

\[ Pr[B|A] = \]

\[ \approx 0.46 \]

\[ \text{fraction of } B \text{ that is inside } A \]
Bayes and Biased Coin

Pick a point uniformly at random in the unit square. Then

\[ Pr[A] = 0.5; \ Pr[\bar{A}] = 0.5 \]
\[ Pr[B|A] = 0.5; \]
Bayes and Biased Coin

Pick a point uniformly at random in the unit square. Then

\[ Pr[A] = 0.5; Pr[\overline{A}] = 0.5 \]
\[ Pr[B|A] = 0.5; Pr[B|\overline{A}] = \]
Bayes and Biased Coin

Pick a point uniformly at random in the unit square. Then

\[ Pr[A] = 0.5; \ Pr[\bar{A}] = 0.5 \]
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Bayes and Biased Coin

Pick a point uniformly at random in the unit square. Then

\[ Pr[A] = 0.5; \ Pr[\bar{A}] = 0.5 \]
\[ Pr[B|A] = 0.5; \ Pr[B|\bar{A}] = 0.6; \ Pr[A \cap B] = \]
Bayes and Biased Coin

Pick a point uniformly at random in the unit square. Then

\[ Pr[A] = 0.5; \Pr[\bar{A}] = 0.5 \]
\[ Pr[B|A] = 0.5; \Pr[B|\bar{A}] = 0.6; \Pr[A \cap B] = 0.5 \times 0.5 \]
Bayes and Biased Coin

Pick a point uniformly at random in the unit square. Then

\[ Pr[A] = 0.5; \ Pr[\bar{A}] = 0.5 \]
\[ Pr[B|A] = 0.5; \ Pr[B|\bar{A}] = 0.6; \ Pr[A \cap B] = 0.5 \times 0.5 \]
\[ Pr[B] = \]
Bayes and Biased Coin

Pick a point uniformly at random in the unit square. Then

\[ Pr[A] = 0.5; \ Pr[\bar{A}] = 0.5 \]
\[ Pr[B|A] = 0.5; \ Pr[B|\bar{A}] = 0.6; \ Pr[A \cap B] = 0.5 \times 0.5 \]
\[ Pr[B] = 0.5 \times 0.5 + 0.5 \times 0.6 \]
Bayes and Biased Coin

Pick a point uniformly at random in the unit square. Then

\[
Pr[A] = 0.5; \quad Pr[\bar{A}] = 0.5
\]

\[
Pr[B|A] = 0.5; \quad Pr[B|\bar{A}] = 0.6; \quad Pr[A \cap B] = 0.5 \times 0.5
\]

\[
Pr[B] = 0.5 \times 0.5 + 0.5 \times 0.6 = Pr[A]Pr[B|A] + Pr[\bar{A}]Pr[B|\bar{A}]
\]
Bayes and Biased Coin

Pick a point uniformly at random in the unit square. Then

\[ Pr[A] = 0.5; \ Pr[\bar{A}] = 0.5 \]
\[ Pr[B|A] = 0.5; \ Pr[B|\bar{A}] = 0.6; \ Pr[A \cap B] = 0.5 \times 0.5 \]
\[ Pr[B] = 0.5 \times 0.5 + 0.5 \times 0.6 = Pr[A]Pr[B|A] + Pr[\bar{A}]Pr[B|\bar{A}] \]
\[ Pr[A|B] = \frac{0.5 \times 0.5}{0.5 \times 0.5 + 0.5 \times 0.6} \]
Bayes and Biased Coin

Pick a point uniformly at random in the unit square. Then

\[ Pr[A] = 0.5; \Pr[\bar{A}] = 0.5 \]

\[ Pr[B|A] = 0.5; \Pr[B|\bar{A}] = 0.6; \Pr[A \cap B] = 0.5 \times 0.5 \]

\[ Pr[B] = 0.5 \times 0.5 + 0.5 \times 0.6 = Pr[A]Pr[B|A] + Pr[\bar{A}]Pr[B|\bar{A}] \]

\[ Pr[A|B] = \frac{0.5 \times 0.5}{0.5 \times 0.5 + 0.5 \times 0.6} = \frac{Pr[A]Pr[B|A]}{Pr[A]Pr[B|A] + Pr[\bar{A}]Pr[B|\bar{A}]} \]

\[ \approx 0.46 \]

fraction of \( B \) that is inside \( A \)
Bayes and Biased Coin

Pick a point uniformly at random in the unit square. Then

\[ Pr[A] = 0.5; \quad Pr[\bar{A}] = 0.5 \]
\[ Pr[B|A] = 0.5; \quad Pr[B|\bar{A}] = 0.6; \quad Pr[A \cap B] = 0.5 \times 0.5 \]
\[ Pr[B] = 0.5 \times 0.5 + 0.5 \times 0.6 = Pr[A]Pr[B|A] + Pr[\bar{A}]Pr[B|\bar{A}] \]
\[ Pr[A|B] = \frac{0.5 \times 0.5}{0.5 \times 0.5 + 0.5 \times 0.6} = \frac{Pr[A]Pr[B|A]}{Pr[A]Pr[B|A] + Pr[\bar{A}]Pr[B|\bar{A}]} \approx 0.46 \]
Bayes and Biased Coin

Pick a point uniformly at random in the unit square. Then

\[
\begin{align*}
Pr[A] &= 0.5; \quad Pr[\bar{A}] = 0.5 \\
Pr[B|A] &= 0.5; \quad Pr[B|\bar{A}] = 0.6; \quad Pr[A \cap B] = 0.5 \times 0.5 \\
Pr[B] &= 0.5 \times 0.5 + 0.5 \times 0.6 = Pr[A]Pr[B|A] + Pr[\bar{A}]Pr[B|\bar{A}] \\
Pr[A|B] &= \frac{0.5 \times 0.5}{0.5 \times 0.5 + 0.5 \times 0.6} = \frac{Pr[A]Pr[B|A]}{Pr[A]Pr[B|A] + Pr[\bar{A}]Pr[B|\bar{A}]}
\end{align*}
\]

\[\approx 0.46 = \text{fraction of } B \text{ that is inside } A\]
Bayes: General Case

Pick a point uniformly at random in the unit square. Then
\[
\Pr[A_m] = p_m, \quad m = 1, \ldots, M
\]
\[
\Pr[B | A_m] = q_m, \quad m = 1, \ldots, M
\]
\[
\Pr[A_m \cap B] = p_m q_m
\]
\[
\Pr[B] = p_1 q_1 + \cdots + p_M q_M
\]
\[
\Pr[A_m | B] = \frac{p_m q_m}{p_1 q_1 + \cdots + p_M q_M}
\]
the fraction of $B$ inside $A_m$. 

Bayes: General Case

Pr[A_m] = p_m, m = 1, ..., M
Pr[B | A_m] = q_m, m = 1, ..., M
Pr[A_m ∩ B] = p_m q_m
Pr[B] = p_1 q_1 + ... + p_M q_M
Pr[A_m | B] = \frac{p_m q_m}{p_1 q_1 + ... + p_M q_M}

Event B
Bayes: General Case

Pick a point uniformly at random in the unit square. Then
Bayes: General Case

Pick a point uniformly at random in the unit square. Then

$$Pr[A_m] = p_m, m = 1, \ldots, M$$
Bayes: General Case

Pick a point uniformly at random in the unit square. Then

\[ Pr[A_m] = p_m, \ m = 1, \ldots, M \]
\[ Pr[B|A_m] = q_m, \ m = 1, \ldots, M; \]
Bayes: General Case

Pick a point uniformly at random in the unit square. Then

\[ Pr[A_m] = p_m, \ m = 1, \ldots, M \]
\[ Pr[B|A_m] = q_m, \ m = 1, \ldots, M; \ Pr[A_m \cap B] = \]
Bayes: General Case

Pick a point uniformly at random in the unit square. Then

\[
Pr[A_m] = p_m, \ m = 1, \ldots, M
\]

\[
Pr[B|A_m] = q_m, \ m = 1, \ldots, M; \ Pr[A_m \cap B] = p_m q_m
\]
Bayes: General Case

Pick a point uniformly at random in the unit square. Then

\[
\begin{align*}
Pr[A_m] &= p_m, \quad m = 1, \ldots, M \\
Pr[B|A_m] &= q_m, \quad m = 1, \ldots, M; \quad Pr[A_m \cap B] = p_m q_m \\
Pr[B] &= p_1 q_1 + \cdots + p_M q_M
\end{align*}
\]
Bayes: General Case

Pick a point uniformly at random in the unit square. Then

\[
\Pr[A_m] = p_m, \quad m = 1, \ldots, M \\
\Pr[B|A_m] = q_m, \quad m = 1, \ldots, M; \quad Pr[A_m \cap B] = p_m q_m \\
\Pr[B] = p_1 q_1 + \cdots + p_M q_M \\
\Pr[A_m|B] = \frac{p_m q_m}{p_1 q_1 + \cdots + p_M q_M}
\]
Bayes: General Case

Pick a point uniformly at random in the unit square. Then

$$Pr[A_m] = p_m, \ m = 1, \ldots, M$$
$$Pr[B|A_m] = q_m, \ m = 1, \ldots, M; \ Pr[A_m \cap B] = p_m q_m$$
$$Pr[B] = p_1 q_1 + \cdots + p_M q_M$$
$$Pr[A_m|B] = \frac{p_m q_m}{p_1 q_1 + \cdots + p_M q_M} = \text{fraction of } B \text{ inside } A_m.$$
Why do you have a fever?

\[
\Pr[\text{Flu} | \text{High Fever}] = 0.15 \times 0.80 = 0.12 \approx 0.58
\]

\[
\Pr[\text{Ebola} | \text{High Fever}] = 10^{-8} \times 1 = 0.00000001 \approx 5 \times 10^{-8}
\]

\[
\Pr[\text{Other} | \text{High Fever}] = 0.85 \times 0.15 \times 0.80 + 10^{-8} \times 1 = 0.42
\]

The values 0.58, 5 \times 10^{-8}, 0.42 are the posterior probabilities.
Why do you have a fever?

Using Bayes’ rule, we find

\[
\begin{align*}
\text{Pr}[	ext{Flu} | \text{High Fever}] &= 0.15 \times 0.80 \times 0.15 \\
\text{Pr}[	ext{Ebola} | \text{High Fever}] &= 10^{-8} \times 1 \\
\text{Pr}[	ext{Other} | \text{High Fever}] &= 0.85 \times 0.1 \times 0.02 \\
\end{align*}
\]

The values 0.58, 5 \times 10^{-8}, 0.42 are the posterior probabilities.
Why do you have a fever?

Using Bayes’ rule, we find

\[
Pr[\text{Flu}|\text{High Fever}] = \frac{0.15 \times 0.80}{0.15 \times 0.80 + 10^{-8} \times 1 + 0.85 \times 0.1} \approx 0.58
\]
Why do you have a fever?

Using Bayes’ rule, we find

\[
Pr[\text{Flu}|\text{High Fever}] = \frac{0.15 \times 0.80}{0.15 \times 0.80 + 10^{-8} \times 1 + 0.85 \times 0.1} \approx 0.58
\]

\[
Pr[\text{Ebola}|\text{High Fever}] = \frac{10^{-8} \times 1}{0.15 \times 0.80 + 10^{-8} \times 1 + 0.85 \times 0.1} \approx 5 \times 10^{-8}
\]
Why do you have a fever?

Using Bayes’ rule, we find

\[ Pr[\text{Flu}|\text{High Fever}] = \frac{0.15 \times 0.80}{0.15 \times 0.80 + 10^{-8} \times 1 + 0.85 \times 0.1} \approx 0.58 \]

\[ Pr[\text{Ebola}|\text{High Fever}] = \frac{10^{-8} \times 1}{0.15 \times 0.80 + 10^{-8} \times 1 + 0.85 \times 0.1} \approx 5 \times 10^{-8} \]

\[ Pr[\text{Other}|\text{High Fever}] = \frac{0.85 \times 0.1}{0.15 \times 0.80 + 10^{-8} \times 1 + 0.85 \times 0.1} \approx 0.42 \]
Why do you have a fever?

Using Bayes’ rule, we find

$$Pr[\text{Flu}|\text{High Fever}] = \frac{0.15 \times 0.80}{0.15 \times 0.80 + 10^{-8} \times 1 + 0.85 \times 0.1} \approx 0.58$$

$$Pr[\text{Ebola}|\text{High Fever}] = \frac{10^{-8} \times 1}{0.15 \times 0.80 + 10^{-8} \times 1 + 0.85 \times 0.1} \approx 5 \times 10^{-8}$$

$$Pr[\text{Other}|\text{High Fever}] = \frac{0.85 \times 0.1}{0.15 \times 0.80 + 10^{-8} \times 1 + 0.85 \times 0.1} \approx 0.42$$

The values 0.58, $5 \times 10^{-8}$, 0.42 are the posterior probabilities.
Why do you have a fever?
Why do you have a fever?

Our “Bayes’ Square” picture:
Why do you have a fever?

Our “Bayes’ Square” picture:

Prior probabilities

\[ \Pr(Fever) = 0.15 \]

\[ \Pr(Ebola) = 10^{-8} \]

\[ \Pr(Other) = 0.85 \]

Conditional probabilities

\[ \Pr(Fever | Flu) = 0.80 \]

\[ \Pr(Fever | Ebola) \approx 0 \]

\[ \Pr(Fever | Other) = 0.10 \]

\[ \frac{58}{100} \text{ of Fever} = \text{Flu} \]

\[ \frac{0}{100} \text{ of Fever} = \text{Ebola} \]

\[ \frac{42}{100} \text{ of Fever} = \text{Other} \]

Green = Fever

Note that even though

\[ \Pr(Fever | Ebola) = 1 \]

one has

\[ \Pr(Ebola | Fever) \approx 0 \]

This example shows the importance of the prior probabilities.
Why do you have a fever?

Our “Bayes’ Square” picture:

\[
\begin{array}{c}
\text{Prior probabilities} \\
0.15 \\
10^{-8} \\
0.85 \\
\text{Other} \\
\text{Conditional probabilities} \\
0.80 \\
1 \\
0.15 \\
0.85 \\
0.10 \\
\text{Green = Fever} \\
\end{array}
\]

\[
\begin{align*}
58\% \text{ of Fever} &= \text{Flu} \\
\approx 0\% \text{ of Fever} &= \text{Ebola} \\
42\% \text{ of Fever} &= \text{Other}
\end{align*}
\]

Note that even though \( Pr[\text{Fever} | \text{Ebola}] = 1 \),
Why do you have a fever?

Our “Bayes’ Square” picture:

Note that even though $Pr[\text{Fever}|\text{Ebola}] = 1$, one has $Pr[\text{Ebola}|\text{Fever}] \approx 0$. 

58% of Fever = Flu
$\approx 0\%$ of Fever = Ebola
42% of Fever = Other
Why do you have a fever?

Our “Bayes’ Square” picture:

Note that even though $Pr[\text{Fever}|\text{Ebola}] = 1$, one has $Pr[\text{Ebola}|\text{Fever}] \approx 0$.

This example shows the importance of the prior probabilities.
Bayes’ Rule Operations
Bayes’ Rule Operations

Bayes’ Rule is the canonical example of how information changes our opinions.

Environment

Priors: $Pr[A_n]$

Observe $B$

Bayes’ Rule

Posterior: $Pr[A_n|B]$

Conditional: $Pr[B|A_n]$

[Model of system]
Bayes’ Rule Operations

Bayes’ Rule is the canonical example of how information changes our opinions.
Recall:

\[ A \text{ and } B \text{ are independent} \]

\[ \Pr[A \cap B] = \Pr[A] \Pr[B] \]

Consider the example below:

\[ \begin{array}{cccccc}
 & A_1 & A_2 & A_3 & \overline{B} & \overline{A_2} \\
A_1 & 0.1 & 0.15 & 0.15 & 0.25 & 0.25 \\
A_2 & 0.15 & 0.1 & 0.15 & 0.25 & 0.1 \\
A_3 & 0.25 & 0.15 & 0.1 & 0.25 & 0.1 \\
\end{array} \]

\((A_2, B)\) are independent:

\[ \Pr[A_2 | B] = \Pr[A_2] \]

\((A_2, \overline{B})\) are independent:

\[ \Pr[A_2 | \overline{B}] = \Pr[A_2] \]

\((A_1, B)\) are not independent:

\[ \Pr[A_1 | B] \neq \Pr[A_1] \]
Independence

Recall:

A and B are independent

\[ \iff Pr[A \cap B] = Pr[A]Pr[B] \]
Independence

Recall:

\( A \) and \( B \) are independent
\( \iff Pr[A \cap B] = Pr[A]Pr[B] \)
\( \iff Pr[A|B] = Pr[A] \).
Independence

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\[ A \text{ and } B \text{ are independent} \iff Pr[A \cap B] = Pr[A]Pr[B] \]
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Consider the example below:
**Independence**

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$\begin{array}{c|c|c}
A_1 & B & \bar{B} \\
\hline
0.1 & 0.15 & 0.15 \\
0.25 & 0.25 & 0.25 \\
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\end{array}$

$(A_2, B)$ are independent:


$Pr[A_2|\bar{B}] = Pr[A_2]$. 

$(A_1, B)$ are not independent:

$Pr[A_1|B] \neq Pr[A_1]$. 

$Pr[A_1|\bar{B}] \neq Pr[A_1]$. 

$Pr[A_1] = 0.25$. 
Independence

Recall:

$A$ and $B$ are independent

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Consider the example below:

$$(A_2, B)$$ are independent: $Pr[A_2|B] = 0.5 = Pr[A_2]$. 

$$(A_2, \bar{B})$$ are independent: $Pr[A_2|\bar{B}] = 0.5 = Pr[A_2]$.

$$(A_1, B)$$ are not independent: $Pr[A_1|B]$ is not equal to $Pr[A_1]$. 

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$$(A_3, B)$$ are not independent: $Pr[A_3|B]$ is not equal to $Pr[A_3]$. 

$$(A_3, \bar{B})$$ are not independent: $Pr[A_3|\bar{B}]$ is not equal to $Pr[A_3]$. 

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Recall:

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\hline
& B & \bar{B} \\
\hline
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A_2 & 0.25 & 0.25 \\
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\hline
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\[(A_2, \bar{B}) \text{ are independent: } Pr[A_2|\bar{B}] = 0.5 = Pr[A_2].\]
\[(A_1, B) \text{ are not independent: } Pr[A_1|B] = \frac{0.1}{0.5} = 0.2 \neq Pr[A_1] = 0.25.\]
Pairwise Independence

Flip two fair coins. Let

- $A = \text{‘first coin is H’} = \{HT, HH\};$
- $B = \text{‘second coin is H’} = \{TH, HH\};$
- $C = \text{‘the two coins are different’} = \{TH, HT\}.$
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$A, C$ are independent; $B, C$ are independent;
$A \cap B, C$ are not independent. $(Pr[A \cap B \cap C] = 0 \neq Pr[A \cap B]Pr[C].)$
Pairwise Independence

Flip two fair coins. Let

- $A = $ ‘first coin is H’ = \{HT, HH\};
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- $C = $ ‘the two coins are different’ = \{TH, HT\}.

$A, C$ are independent; $B, C$ are independent; $A \cap B, C$ are not independent. ($Pr[A \cap B \cap C] = 0 \neq Pr[A \cap B]Pr[C].$)

$A$ did not say anything about $C$ and $B$ did not say anything about $C$, but $A \cap B$ said something about $C$!
Example 2

Flip a fair coin 5 times.
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Flip a fair coin 5 times. Let $A_n = ‘\text{coin } n \text{ is H}’$, for $n = 1, \ldots, 5$. 
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$$A_m, A_n \text{ are independent for all } m \neq n.$$
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Also,

$A_1$ and $A_3 \cap A_5$ are independent.
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Indeed,

$$Pr[A_1 \cap (A_3 \cap A_5)] = \frac{1}{8} = Pr[A_1]Pr[A_3 \cap A_5]$$
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. Similarly,

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Example 2

Flip a fair coin 5 times. Let $A_n = \text{\textquoteleft coin } n \text{ is H\textquoteright}$, for $n = 1, \ldots, 5$.

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Indeed,

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Similarly,

$A_1 \cap A_2$ and $A_3 \cap A_4 \cap A_5$ are independent.

This leads to a definition ....
Definition Mutual Independence

(a) The events $A_1, \ldots, A_5$ are mutually independent if

$$\Pr \left[ \bigcap_{k \in K} A_k \right] = \prod_{k \in K} \Pr[A_k],$$

for all $K \subseteq \{1, \ldots, 5\}$.

(b) More generally, the events $\{A_j, j \in J\}$ are mutually independent if

$$\Pr \left[ \bigcap_{k \in K} A_k \right] = \prod_{k \in K} \Pr[A_k],$$

for all finite $K \subseteq J$.

Example: Flip a fair coin forever. Let $A_n = \text{`coin n is H.'}$ Then the events $A_n$ are mutually independent.
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Example: Flip a fair coin forever. Let $A_n = \text{‘coin } n \text{ is H’}$. Then the events $A_n$ are mutually independent.
Mutual Independence

**Theorem**

(a) If the events \{A_j, j \in J\} are mutually independent and if \(K_1\) and \(K_2\) are disjoint finite subsets of \(J\), then \(\bigcap_{k \in K_1} A_k\) and \(\bigcap_{k \in K_2} A_k\) are independent.

(b) More generally, if the \(K_n\) are pairwise disjoint finite subsets of \(J\), then the events \(\bigcap_{k \in K_n} A_k\) are mutually independent.

(c) Also, the same is true if we replace some of the \(A_k\) by \(\bar{A}_k\).
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Balls in bins

One throws $m$ balls into $n > m$ bins.
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Theorem: $\Pr[\text{no collision}] \approx \exp\left\{-\frac{m^2}{2n}\right\}$, for large enough $n$. 

$Pr[\text{bin } k] = \frac{1}{n}$ for $k = 1, \ldots, n$
Balls in bins

One throws $m$ balls into $n > m$ bins.

**Theorem:**

$$Pr[\text{no collision}] \approx \exp\left\{-\frac{m^2}{2n}\right\}, \text{ for large enough } n.$$
The Calculation.

$A_i = \text{no collision when } i\text{th ball is placed in a bin.}$
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\[ Pr[A_1] = 1 \]

\[ Pr[A_2|A_1] = \]

\[ Pr[A_1 \cap \cdots \cap A_m] = \left(1 - \frac{1}{n}\right) \cdots \left(1 - \frac{m}{n}\right). \]
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no collision = \( A_1 \cap \cdots \cap A_m \).
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\[Pr[A_3|A_1, A_2] = 1 - \frac{2}{n}\]

\[Pr[A_i|A_{i-1} \cap \cdots \cap A_1] = \left(1 - \frac{i-1}{n}\right)\]

no collision = \(A_1 \cap \cdots \cap A_m\).

Product rule:
The Calculation.

$A_i =$ no collision when $i$th ball is placed in a bin.

$Pr[A_1] = 1$

$Pr[A_2|A_1] = 1 - \frac{1}{n}$

$Pr[A_3|A_1, A_2] = 1 - \frac{2}{n}$

$Pr[A_i|A_{i-1} \cap \cdots \cap A_1] = (1 - \frac{i-1}{n})$.

no collision $= A_1 \cap \cdots \cap A_m$.

Product rule:

$Pr[A_1 \cap \cdots \cap A_m] = Pr[A_1] Pr[A_2|A_1] \cdots Pr[A_m|A_1 \cap \cdots \cap A_{m-1}]$
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\[ A_i = \text{no collision when } i\text{th ball is placed in a bin.} \]

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\[ \Rightarrow Pr[\text{no collision}] = \left(1 - \frac{1}{n}\right) \cdots \left(1 - \frac{m-1}{n}\right). \]

Hence,

\[ \ln(Pr[\text{no collision}]) = \sum_{k=1}^{m-1} \ln\left(1 - \frac{k}{n}\right) \]

\[ \approx \frac{1}{n} m \left(m - 1\right) \quad \text{(\* \*)} \]
\[ \Rightarrow Pr[\text{no collision}] = \left(1 - \frac{1}{n}\right) \cdots \left(1 - \frac{m-1}{n}\right). \]

Hence,

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\ln(Pr[\text{no collision}]) = \sum_{k=1}^{m-1} \ln \left(1 - \frac{k}{n}\right) \approx \sum_{k=1}^{m-1} \left(-\frac{k}{n}\right) \quad (*)
\]
⇒ $Pr[\text{no collision}] = \left(1 - \frac{1}{n}\right) \cdots \left(1 - \frac{m-1}{n}\right)$.

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$\Rightarrow Pr[\text{no collision}] = \left(1 - \frac{1}{n}\right) \cdots \left(1 - \frac{m-1}{n}\right)$.

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⇒ \( Pr[\text{no collision}] = \left(1 - \frac{1}{n}\right) \cdots \left(1 - \frac{m-1}{n}\right). \)

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\[
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\]

(**) We used \( \ln(1 - \varepsilon) \approx -\varepsilon \) for \( |\varepsilon| \ll 1. \)
\[ \Rightarrow Pr[\text{no collision}] = \left(1 - \frac{1}{n}\right) \cdots \left(1 - \frac{m-1}{n}\right). \]

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\]

\[= -\frac{1}{n} \frac{m(m-1)}{2} \approx -\frac{m^2}{2n} \quad (†)
\]

\((*)\) We used \(\ln(1 - \varepsilon) \approx -\varepsilon\) for \(|\varepsilon| \ll 1\).

\((†)\) \(1 + 2 + \cdots + m-1 = (m-1)m/2\).
Approximation

\[
\exp\{-x\} = 1 - x + \frac{1}{2!} x^2 + \cdots \approx 1 - x, \quad \text{for } |x| \ll 1.
\]

Hence, \(-x \approx \ln(1 - x)\) for \(|x| \ll 1\).
Balls in bins

**Theorem:**

\[ Pr[\text{no collision}] \approx \exp\left\{ -\frac{m^2}{2n} \right\}, \text{ for large enough } n. \]
Balls in bins

Theorem:

\[ Pr[\text{no collision}] \approx \exp\left\{-\frac{m^2}{2n}\right\} \], for large enough \( n \).

E.g., with \( m = 6 \) one has

\[ Pr[\text{collision}] > 1/2 \]
Theorem:
$$Pr[\text{no collision}] \approx \exp\left\{-\frac{m^2}{2n}\right\}, \text{ for large enough } n.$$
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**Theorem:**

\[ Pr[\text{no collision}] \approx \exp\{-\frac{m^2}{2n}\}, \text{ for large enough } n. \]

In particular, \( Pr[\text{no collision}] \approx 1/2 \) for \( m^2/(2n) \approx \ln(2) \), i.e.,

\[ m \approx \sqrt{2\ln(2)n} \approx 1.2\sqrt{n}. \]
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E.g., \( 1.2\sqrt{20} \approx 5.4. \)
Balls in bins

**Theorem:**

\[ Pr[\text{no collision}] \approx \exp\left\{-\frac{m^2}{2n}\right\}, \text{ for large enough } n. \]

In particular, \( Pr[\text{no collision}] \approx 1/2 \) for \( m^2/(2n) \approx \ln(2) \), i.e.,

\[ m \approx \sqrt{2\ln(2)n} \approx 1.2\sqrt{n}. \]

E.g., \( 1.2\sqrt{20} \approx 5.4 \).

Roughly, \( Pr[\text{collision}] \approx 1/2 \) for \( m = \sqrt{n} \).
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Roughly, \( Pr[\text{collision}] \approx \frac{1}{2} \) for \( m = \sqrt{n} \). \( e^{-0.5} \approx 0.6 \).
The birthday paradox
Today’s your birthday, it’s my birthday too..

Probability that $m$ people all have different birthdays?
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If \( m = 366 \), then \( Pr[\text{no collision}] = 0 \). (No approximation here!)
The birthday paradox

<table>
<thead>
<tr>
<th>$n$</th>
<th>$p(n)$</th>
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</thead>
<tbody>
<tr>
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<tr>
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<td>2.7%</td>
</tr>
<tr>
<td>10</td>
<td>11.7%</td>
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<tr>
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</tr>
<tr>
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<td>70.6%</td>
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<tr>
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<td>89.1%</td>
</tr>
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<td>97.0%</td>
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</tr>
<tr>
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<td>99.9997%</td>
</tr>
<tr>
<td>200</td>
<td>99.99999999999999999999999999998%</td>
</tr>
<tr>
<td>300</td>
<td>$(100 - (6 \times 10^{-80}))%$</td>
</tr>
<tr>
<td>350</td>
<td>$(100 - (3 \times 10^{-129}))%$</td>
</tr>
<tr>
<td>365</td>
<td>$(100 - (1.45 \times 10^{-155}))%$</td>
</tr>
<tr>
<td>366</td>
<td>100%</td>
</tr>
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<td>100%</td>
</tr>
</tbody>
</table>
Checksums!

Consider a set of $m$ files. Each file has a checksum of $b$ bits. How large should $b$ be for $\Pr[\text{share a checksum}] \leq 10^{-3}$?

Claim: $b \geq 2.9 \ln(m) + 9.2$.

Proof: Let $n = 2^b$ be the number of checksums. We know $\Pr[\text{no collision}] \approx \exp\left(-\frac{m^2}{2n}\right) \approx 1 - \frac{m^2}{2n}$.

Hence, $\Pr[\text{no collision}] \approx 1 - 10^{-3} \iff \frac{m^2}{2n} \approx 10^{-3} \iff 2n \approx m^2 10^{-3} \iff 2^b + 1 \approx m^2 10^{-3} \iff b + 1 \approx 10 + 2\log_2(m) \approx 10 + 2 \cdot 1.44 \ln(m)$.

Note: $\log_2(x) = \log_2(e) \ln(x) \approx 1.44 \ln(x)$. 
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Note: $\log_2(x) = \log_2(e) \ln(x) \approx 1.44 \ln(x)$. 

Coupon Collector Problem.

There are $n$ different baseball cards.
(Brian Wilson, Jackie Robinson, Roger Hornsby, ...)

Theorem:
(a) $\Pr[\text{miss one specific item}] \approx e^{-m/n}$
(b) $\Pr[\text{miss any one of the items}] \leq n e^{-m/n}$. 
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Coupon Collector Problem: Analysis.

Event $A_m = \text{‘fail to get Brian Wilson in } m \text{ cereal boxes’}$
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Event $A_m = \text{‘fail to get Brian Wilson in } m \text{ cereal boxes’}$

Fail the first time: $(1 - \frac{1}{n})$
Event $A_m$ = ‘fail to get Brian Wilson in $m$ cereal boxes’
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Fail the second time: $(1 - \frac{1}{n})$
Event $A_m = \text{‘fail to get Brian Wilson in } m \text{ cereal boxes’}

Fail the first time: $(1 - \frac{1}{n})$

Fail the second time: $(1 - \frac{1}{n})$

And so on ...
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And so on ... for $m$ times. Hence,

$$Pr[A_m] = (1 - \frac{1}{n}) \times \cdots \times (1 - \frac{1}{n})$$
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For $p_m = \frac{1}{2}$, we need around $n \ln 2 \approx 0.69n$ boxes.
Collect all cards?

Experiment: Choose $m$ cards at random with replacement.
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Events: $E_k = \text{‘fail to get player } k\text{’}$, for $k = 1, \ldots, n$
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Probability of failing to get at least one of these $n$ players:

$$p := Pr[E_1 \cup E_2 \cdots \cup E_n]$$
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How does one estimate \( p \)? **Union Bound:**

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p = Pr[E_1 \cup E_2 \cdots \cup E_n] \leq Pr[E_1] + Pr[E_2] \cdots Pr[E_n].
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\Pr[E_k] \approx e^{-\frac{m}{n}}, k = 1, \ldots, n.
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Plug in and get

$$p \leq ne^{-\frac{m}{n}}.$$
Collect all cards?

Thus,

$$Pr[\text{missing at least one card}] \leq ne^{-\frac{m}{n}}.$$
Collect all cards?

Thus,

\[ P_r[\text{missing at least one card}] \leq ne^{-\frac{m}{n}}. \]

Hence,

\[ P_r[\text{missing at least one card}] \leq p \text{ when } m \geq n \ln\left(\frac{n}{p}\right). \]
To collect all cards?

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E.g., \( n = 10^2 \Rightarrow m = 530; \)
Collect all cards?

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Summary.

Bayes’ Rule, Mutual Independence, Collisions and Collecting
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Main results:

- **Bayes’ Rule**: \( \Pr[A_m|B] = \frac{p_m q_m}{(p_1 q_1 + \cdots + p_M q_M)} \).
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  \Pr[A_1 \cap \cdots \cap A_n] = \Pr[A_1] \Pr[A_2|A_1] \cdots \Pr[A_n|A_1 \cap \cdots \cap A_{n-1}].
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Bayes’ Rule, Mutual Independence, Collisions and Collecting

Main results:

► **Bayes’ Rule:** \( Pr[A_m | B] = p_m q_m / (p_1 q_1 + \cdots + p_M q_M) \).

► **Product Rule:**
\[
Pr[A_1 \cap \cdots \cap A_n] = Pr[A_1] Pr[A_2 | A_1] \cdots Pr[A_n | A_1 \cap \cdots \cap A_{n-1}].
\]

► **Balls in bins:** \( m \) balls into \( n > m \) bins.
\[
Pr[\text{no collisions}] \approx \exp\{-\frac{m^2}{2n}\}
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► **Coupon Collection:** \( n \) items. Buy \( m \) cereal boxes.
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Pr[\text{miss one specific item}] \approx e^{-\frac{m}{n}};
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Summary.

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- **Bayes’ Rule:** $Pr[A_m|B] = \frac{p_m q_m}{(p_1 q_1 + \cdots + p_M q_M)}$.
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Key Mathematical Fact: $\ln(1 - \varepsilon) \approx -\varepsilon$. 