Modeling Uncertainty: Probability Space
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1. Key Points
2. Random Experiments
3. Probability Space
4. Events
Key Points

Uncertainty does not mean “nothing is known”

How to best make decisions under uncertainty?

Buy stocks
Detect signals (transmitted bits, speech, images, radar, diseases, etc.)
Control systems (Internet, airplane, robots, self-driving cars, schedule surgeries in a hospital, etc.)

How to best use ‘artificial’ uncertainty?

Play games of chance.
Design randomized algorithms.
Catch Pokemon.

Probability

Models knowledge about uncertainty
Discovers best way to use that knowledge in making decisions
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  - Models knowledge about uncertainty
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  ▶ Discovers best way to use that knowledge in making decisions
The Magic of Probability

Uncertainty: vague, fuzzy, confusing, scary, hard to think about.

Probability: a precise, unambiguous, simple way to think about uncertainty.

Our mission: help you discover the magic of Probability, i.e., enable you to think clearly about uncertainty.

Your cost: focused attention and practice on examples and problems.
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**Uncertainty**: vague, fuzzy, confusing, scary, hard to think about.

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![The Scream by Edvard Munch]

Uncertainty = Fear
The Magic of Probability

**Uncertainty:** vague, fuzzy, confusing, scary, hard to think about.

**Probability:** A precise, unambiguous, simple way to think about uncertainty.

Uncertainty = Fear

Probability = Serenity
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Uncertainty = Fear

Probability = Serenity

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Your cost: focused attention and practice on examples and problems.
A cool trick
Random Experiment: Flip one Fair Coin

Flip a fair coin:

▶ Possible outcomes: Heads (H) and Tails (T)

▶ Likelihoods: H: 50% and T: 50%
Random Experiment: Flip one Fair Coin

Flip a fair coin:
Random Experiment: Flip one Fair Coin

Flip a fair coin: (*One flips or tosses a coin*)
Random Experiment: Flip one Fair Coin

Flip a *fair* coin: (*One flips or tosses a coin*)

![Image of a coin being flipped]

Possible outcomes: Heads (H) and Tails (T)

Likelihoods:
- H: 50%
- T: 50%
Random Experiment: Flip one Fair Coin

Flip a fair coin: *(One flips or tosses a coin)*

- Possible outcomes:
Random Experiment: Flip one Fair Coin

Flip a **fair** coin: *(One flips or tosses a coin)*

- **Possible outcomes:** Heads *(H)*
Random Experiment: Flip one Fair Coin

Flip a **fair** coin: *(One flips or tosses a coin)*

- Possible outcomes: Heads \((H)\) and Tails \((T)\)
Random Experiment: Flip one Fair Coin

Flip a fair coin: \((One \ flips \ or \ tosses \ a \ coin)\)

- Possible outcomes: Heads \((H)\) and Tails \((T)\)
  \((One \ flip \ yields \ either \ ‘heads’ \ or \ ‘tails’.)\)
Random Experiment: Flip one Fair Coin

Flip a *fair* coin: (*One flips or tosses a coin*)

- Possible outcomes: Heads (H) and Tails (T)
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- Likelihoods:
Random Experiment: Flip one Fair Coin

Flip a fair coin: (One flips or tosses a coin)

- Possible outcomes: Heads (H) and Tails (T) (One flip yields either ‘heads’ or ‘tails’.)
- Likelihoods: $H: 50\%$ and $T: 50\%$
Random Experiment: Flip one Fair Coin

Flip a fair coin:

What do we mean by the likelihood of tails is 50%?

Two interpretations:

▶ Single coin flip: 50% chance of 'tails'
▶ Many coin flips: About half yield 'tails'

Makes sense for many flips

▶ Question:

Why does the fraction of tails converge to the same value every time?

Statistical Regularity!
Random Experiment: Flip one Fair Coin

Flip a **fair** coin:

What do we mean by **the likelihood of tails is 50%**?

Two interpretations:
Random Experiment: Flip one Fair Coin

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  *Makes sense for many flips*

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Random Experiment: Flip one Fair Coin

Flip a fair coin:
Random Experiment: Flip one Fair Coin

Flip a **fair** coin: model
Random Experiment: Flip one Fair Coin

Flip a *fair* coin: model

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**Physical Experiment**

**Probability Model**

- $\Omega = \{H, T\}$
- $\Pr[H] = 0.5$
- $\Pr[T] = 0.5$
Random Experiment: Flip one Fair Coin

Flip a fair coin: model

The physical experiment is complex.
Random Experiment: Flip one Fair Coin

Flip a fair coin: model

- The physical experiment is complex. (Shape, density, initial momentum and position, ...)

Ω

H 0.5

T 0.5
Random Experiment: Flip one Fair Coin

Flip a fair coin: model

- The physical experiment is complex. (Shape, density, initial momentum and position, ...)
- The Probability model is simple:
Random Experiment: Flip one Fair Coin

Flip a fair coin: model

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- The Probability model is simple:
  - A set $\Omega$ of outcomes: $\Omega = \{H, T\}$.
Random Experiment: Flip one Fair Coin

Flip a fair coin: model

- The physical experiment is complex. (Shape, density, initial momentum and position, ...)
- The Probability model is simple:
  - A set $\Omega$ of outcomes: $\Omega = \{H, T\}$.
  - A probability assigned to each outcome: $Pr[H] = 0.5, Pr[T] = 0.5$.  

- **Physical Experiment**
- **Probability Model**
Random Experiment: Flip one Unfair Coin

Flip an unfair (biased, loaded) coin:

Possible outcomes: Heads (H) and Tails (T)

Likelihoods:
- H: \( p \in (0, 1) \)
- T: \( 1 - p \)

Frequentist Interpretation:
Flip many times ⇒ Fraction \( 1 - p \) of tails

Question:
How can one figure out \( p \)?
Flip many times
Random Experiment: Flip one Unfair Coin

Flip an unfair (biased, loaded) coin:
Random Experiment: Flip one Unfair Coin

Flip an *unfair* (biased, loaded) coin:

H: 45%
T: 55%
Random Experiment: Flip one Unfair Coin

Flip an unfair (biased, loaded) coin:

Possible outcomes:

- H: 45%
- T: 55%
Random Experiment: Flip one Unfair Coin

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Random Experiment: Flip one Unfair Coin

Flip an **unfair** (biased, loaded) coin:

- Possible outcomes: Heads ($H$) and Tails ($T$)
- Likelihoods: $H : p \in (0, 1)$ and $T : 1 - p$

![Unfair coin with 45% Heads and 55% Tails]
Random Experiment: Flip one Unfair Coin

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- Question: How can one figure out $p$? Flip many times

H: 45%  
T: 55%
Random Experiment: Flip one Unfair Coin

Flip an unfair (biased, loaded) coin:

- Possible outcomes: Heads \( (H) \) and Tails \( (T) \)
- Likelihoods: \( H : p \in (0, 1) \) and \( T : 1 - p \)
- Frequentist Interpretation:
  
  Flip many times \( \Rightarrow \) Fraction \( 1 - p \) of tails

- Question: How can one figure out \( p \)? Flip many times
- Tautology?
Random Experiment: Flip one Unfair Coin
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Flip an *unfair* (biased, loaded) coin: model
Random Experiment: Flip one Unfair Coin

Flip an **unfair** (biased, loaded) coin: model

- **Physical Experiment**
- **Probability Model**

Same set of outcomes as before!
Different probabilities!

The most common mistake in Probability: assuming that outcomes are equally likely.
Random Experiment: Flip one Unfair Coin

Flip an unfair (biased, loaded) coin: model

- Same set of outcomes as before!
Random Experiment: Flip one Unfair Coin

Flip an **unfair** (biased, loaded) coin: model

▶ Same set of outcomes as before!
▶ Different probabilities!

Physical Experiment

\[
\begin{align*}
\Omega &= \{H, T\} \\
H \cap p &\quad T \cap (1 - p)
\end{align*}
\]
Random Experiment: Flip one Unfair Coin

Flip an **unfair** (biased, loaded) coin: model

- **Same set of outcomes as before!**
- **Different probabilities!**
- **The most common mistake in Probability:**
Random Experiment: Flip one Unfair Coin

Flip an unfair (biased, loaded) coin: model

- Same set of outcomes as before!
- Different probabilities!
- The most common mistake in Probability: assuming that outcomes are equally likely.
Flip Two Fair Coins

Possible outcomes: \{HH, HT, TH, TT\} = \{H, T\}^2.

Note: \(A \times B := \{(a, b) \mid a \in A, b \in B\}\) and \(A^2 := A \times A\).

Likelihoods: \(1/4\) each.
Flip Two Fair Coins

Possible outcomes:

Possible outcomes include:

- HH
- HT
- TH
- TT

Each outcome has a likelihood of 1/4.
Flip Two Fair Coins

- Possible outcomes: \( \{HH, HT, TH, TT\} \)
Flip Two Fair Coins

Possible outcomes: \{HH, HT, TH, TT\} \equiv \{H, T\}^2.
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- Likelihoods: 1/4 each.
Flip Glued Coins

Possible outcomes: {HH, TT}.

Likelihoods: HH: 0.5, TT: 0.5.

Note: Coins are glued so that they show the same face.
Flip Glued Coins

Flips two coins glued together side by side:
Flip Glued Coins

Flips two coins glued together side by side:

- **Possible outcomes:** {HH, TT}
- **Likelihoods:**
  - HH: 0.5
  - TT: 0.5

Note: Coins are glued so that they show the same face.
Flip Glued Coins

Flips two coins glued together side by side:

- Possible outcomes: \{HH, TT\}

- Likelihoods:
  - HH: 0.5
  - TT: 0.5

Note: Coins are glued so that they show the same face.
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Flips two coins glued together side by side:

- Possible outcomes: \{HH, TT\}.
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Flips two coins glued together side by side:

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- Likelihoods:
  - HH: 0.5
  - TT: 0.5

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Flips two coins glued together side by side:

- Possible outcomes: \{HH, TT\}.
- Likelihoods: HH : 0.5, TT : 0.5.
Flip Glued Coins

Flips two coins glued together side by side:

- Possible outcomes: \( \{HH, TT\} \).
- Likelihoods: \( HH : 0.5, TT : 0.5 \).
- Note: Coins are glued so that they show the same face.
Flip Glued Coins

Flips two coins glued together side by side:

▶ Possible outcomes: {HT, TH}.

▶ Likelihoods: HT: 0.5, TH: 0.5.

Note: Coins are glued so that they show different faces.
Flip Glued Coins

Flips two coins glued together side by side:

Possible outcomes: {HT, TH}.

Likelihoods: HT: 0.5, TH: 0.5.

Note: Coins are glued so that they show different faces.
Flip Glued Coins

Flips two coins glued together side by side:

Possible outcomes: {HT, TH}.

Likelihoods: HT: 0.5, TH: 0.5.

Note: Coins are glued so that they show different faces.
Flip Glued Coins

Flips two coins glued together side by side:

Possible outcomes:

- HT: 50%
- TH: 50%
Flip Glued Coins

Flips two coins glued together side by side:

Possible outcomes: \{HT, TH\}.
Flip Glued Coins

Flips two coins glued together side by side:

- Possible outcomes: \( \{HT, TH\} \).
- Likelihoods:

  - \( HT \): 0.5
  - \( TH \): 0.5
Flip Glued Coins

Flips two coins glued together side by side:

- Possible outcomes: \{HT, TH\}.
- Likelihoods: HT : 0.5, TH : 0.5.
Flip Glued Coins

Flips two coins glued together side by side:

- Possible outcomes: \{HT, TH\}.
- Likelihoods: $HT : 0.5$, $TH : 0.5$.
- Note: Coins are glued so that they show different faces.
Flip two Attached Coins

Possible outcomes:

- HH
- HT
- TH
- TT

Likelihoods:

- HH: 0.4
- HT: 0.1
- TH: 0.1
- TT: 0.4

Note: Coins are attached so that they tend to show the same face, unless the spring twists enough.
Flip two Attached Coins

Flips two coins attached by a spring:

Possible outcomes:

- HH
- HT
- TH
- TT

Likelihoods:

- HH: 0.4
- HT: 0.1
- TH: 0.1
- TT: 0.4

Note: Coins are attached so that they tend to show the same face, unless the spring twists enough.
Flip two Attached Coins

Flips two coins attached by a spring:

Possible outcomes:
- HH
- HT
- TH
- TT

Likelihoods:
- HH: 0.4
- HT: 0.1
- TH: 0.1
- TT: 0.4

Note: Coins are attached so that they tend to show the same face, unless the spring twists enough.
Flip two Attached Coins

Flips two coins attached by a spring:

- Possible outcomes:
  - HH: 0.4
  - HT: 0.1
  - TH: 0.1
  - TT: 0.4

Note: Coins are attached so that they tend to show the same face, unless the spring twists enough.
Flip two Attached Coins

Flips two coins attached by a spring:

Possible outcomes: \( \{HH, HT, TH, TT\} \).
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Flips two coins attached by a spring:

Possible outcomes: \( \{ HH, HT, TH, TT \} \).

Likelihoods:
Flip two Attached Coins

Flips two coins attached by a spring:

Possible outcomes: \{HH, HT, TH, TT\}.

Likelihoods: HH : 0.4, HT : 0.1, TH : 0.1, TT : 0.4.
Flip two Attached Coins

Flips two coins attached by a spring:

Possible outcomes: \{HH, HT, TH, TT\}.

Likelihoods: \text{HH} : 0.4, \text{HT} : 0.1, \text{TH} : 0.1, \text{TT} : 0.4.

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Flipping Two Coins

Here is a way to summarize the four random experiments:

- \( \Omega \) is the set of possible outcomes;
- Each outcome has a probability (likelihood);
- The probabilities are \( \geq 0 \) and add up to 1;
- Fair coins: \([1]\);
- Glued coins: \([3], [4]\);
- Spring-attached coins: \([2]\);
Flipping Two Coins

Here is a way to summarize the four random experiments:

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- Spring-attached coins: \[2\].
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Here is a way to summarize the four random experiments:

1. $\Omega$ is the set of possible outcomes; each outcome has a probability (likelihood);
2. The probabilities are $\geq 0$ and add up to 1;

\begin{align*}
\Omega & : TH \quad HH \\
 & : 0.25 \quad 0.25 \\
TT & : 0.25 \quad HT \\
\end{align*}

\begin{align*}
\Omega & : TH \quad HH \\
 & : 0.1 \quad 0.4 \\
TT & : 0.4 \quad HT \\
\end{align*}

\begin{align*}
\Omega & : TH \quad HH \\
 & : 0 \quad 0.5 \\
TT & : 0.5 \quad HT \\
\end{align*}

\begin{align*}
\Omega & : TH \quad HH \\
 & : 0.5 \quad 0 \\
TT & : 0 \quad 0.5 \\
\end{align*}
Flipping Two Coins

Here is a way to summarize the four random experiments:

1. $\Omega$ is the set of possible outcomes;

   - Fair coins: $\left[ \begin{array}{cc} TH & HH \\ 0.25 & 0.25 \end{array} \right]$; $\left[ \begin{array}{cc} TT & HT \\ 0.25 & 0.25 \end{array} \right]$;
   - Glued coins: $\left[ \begin{array}{cc} TH & HH \\ 0.1 & 0.4 \end{array} \right]$; $\left[ \begin{array}{cc} TT & HT \\ 0.4 & 0.1 \end{array} \right]$;
   - Spring-attached coins: $\left[ \begin{array}{cc} TH & HH \\ 0 & 0.5 \end{array} \right]$; $\left[ \begin{array}{cc} TT & HT \\ 0.5 & 0 \end{array} \right]$;
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Here is a way to summarize the four random experiments:

Important remarks:

▶ Each outcome describes the two coins.
▶ E.g., "HT" is one outcome of the experiment.
▶ It is wrong to think that the outcomes are \{H, T\} and that one picks twice from that set.
▶ This viewpoint misses the relationship between the two flips.
▶ Each \( \omega \in \Omega \) describes one outcome of the complete experiment.
▶ \( \Omega \) and the probabilities specify the random experiment.
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Flip a fair coin $n$ times (some $n \geq 1$):
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- Possible outcomes: $\{TT \cdots T, TT \cdots H, \ldots, HH \cdots H\}$. 

Thus, $2^n$ possible outcomes.

Note: $\{TT \cdots T, TT \cdots H, \ldots, HH \cdots H\} = \{T, H\}^n$.

\[ A_n = \{ (a_1, \ldots, a_n) | a_1 \in A, \ldots, a_n \in A \} \]

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Flip a **fair** coin $n$ times (some $n \geq 1$):

- **Possible outcomes:** $\{TT \cdots T, TT \cdots H, \ldots, HH \cdots H\}$. Thus, $2^n$ possible outcomes.
- **Note:** $\{TT \cdots T, TT \cdots H, \ldots, HH \cdots H\} = \{H, T\}^n$.
  
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Roll two Dice

Roll a balanced 6-sided die twice:
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Roll a *balanced* 6-sided die twice:

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Roll a balanced 6-sided die twice:

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  \[ \{1, 2, 3, 4, 5, 6\}^2 = \{(a, b) \mid 1 \leq a, b \leq 6\}. \]
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Probability Space.

1. A “random experiment”:

(a) Flip a biased coin;
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2. A set of possible outcomes: \( \Omega \).

(a) \( \Omega = \{H, T\} \);
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(c) \( \Omega = \{A♠A♦A♣A♥K♠, A♠A♦A♣A♥Q♠, \ldots\} \); \(|\Omega| = \binom{52}{5} \).

3. Assign a probability to each outcome: \( \Pr: \Omega \to [0, 1] \).

(a) \( \Pr[H] = p, \Pr[T] = 1 - p \) for some \( p \in [0, 1] \);
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In a **uniform probability space** each outcome $\omega$ is **equally probable**: $Pr[\omega] = \frac{1}{|\Omega|}$ for all $\omega \in \Omega$. 

Examples: ▶ Flipping two fair coins, dealing a poker hand are uniform probability spaces. ▶ Flipping a biased coin is not a uniform probability space.
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Probability Space: Formalism

Simplest physical model of a **uniform** probability space:

A bag of identical balls, except for their color (or a label). If the bag is well shaken, every ball is equally likely to be picked. $\Omega = \{\text{white, red, yellow, grey, purple, blue, maroon, green}\}$

$\Pr[\text{blue}] = \frac{1}{8}$. 
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\[ Pr[\omega] = \]

\[ \begin{array}{c}
\text{Red} \\
\text{Green} \\
\text{Maroon} \\
\vdots \\
\end{array} \]

\[ 1/8 \]

\[ ... \]

\[ 1/8 \]
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Probability Space: Formalism

Simplest physical model of a non-uniform probability space:
Probability Space: Formalism

Simplest physical model of a **non-uniform** probability space:

\[ \Omega = \{ \text{Red}, \text{Green}, \text{Yellow}, \text{Blue} \} \]

\[ \Pr[\text{Red}] = \frac{3}{10}, \quad \Pr[\text{Green}] = \frac{4}{10}, \quad \text{etc.} \]

Note: Probabilities are restricted to rational numbers: \( \mathbb{Q} \).
Probability Space: Formalism

Simplest physical model of a non-uniform probability space:

\[ \Omega = \{ \text{Red, Green, Yellow, Blue} \} \]

Physical experiment

![Physical experiment diagram]

Probability model

\[ \Omega \]

\[ \Pr[\omega] \]

- Red: 3/10
- Green: 4/10
- Yellow: 2/10
- Blue: 1/10

Note: Probabilities are restricted to rational numbers: \( \frac{N}{k} \) ∈ \( \mathbb{N} \).
Probability Space: Formalism

Simplest physical model of a non-uniform probability space:

$$\Omega = \{\text{Red, Green, Yellow, Blue}\}$$

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Probability Space: Formalism

Physical model of a general non-uniform probability space:
Probability Space: Formalism

Physical model of a general non-uniform probability space:

![Diagram of a probability space](image)

The roulette wheel stops in sector $\omega$ with probability $p_\omega$.

$\Omega = \{1, 2, 3, \ldots, N\}$, $Pr[\omega] = p_\omega$. 

- Green $= 1$
- Purple $= 2$
- Yellow

Physical experiment

Probability model
Probability Space: Formalism

Physical model of a general non-uniform probability space:

The roulette wheel stops in sector $\omega$ with probability $p_\omega$. 

Physical experiment  Probability model

Green = 1
Purple = 2
Yellow $\omega$

$\Omega = \{1, 2, 3, \ldots, N\}$,
$\Pr[\omega] = p_\omega$. 

\[ \text{Fraction } p_1 \text{ of circumference} \]

\[ \omega \]

\[ p_1 \]

\[ p_2 \]

\[ \ldots \]

\[ p_\omega \]
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An important remark

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- For instance, say we glue the coins side-by-side so that they face up the same way. Then one gets $HH$ or $TT$ with probability 50% each. This is not captured by ‘picking two outcomes.’
Events

Next idea: an event!
Set notation review
Set notation review

Figure: Two events
Set notation review

\[ \Omega \]

\[ A \quad B \]

**Figure : Two events**

\[ \Omega \]

\[ \bar{A} \]

**Figure : Complement (not)**
Set notation review

Figure: Two events

Figure: Complement (not)

Figure: Union (or)
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Set notation review

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Figure: Difference (A, not B)

Figure: Complement (not)

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Set notation review

- $A \cap B$: Intersection (and)
- $A \cup B$: Union (or)
- $A \setminus B$: Difference ($A$, not $B$)
- $A^\complement$: Complement (not)
- $A \Delta B$: Symmetric difference (only one)

**Figure:** Two events
Probability of exactly one ‘heads’ in two coin flips?

Idea: Sum the probabilities of all the different outcomes that have exactly one ‘heads’:

HT, TH.

This leads to a definition!

Definition:

▶ An event, \( E \), is a subset of outcomes: \( E \subset \Omega \).
▶ The probability of \( E \) is defined as \( \Pr[E] = \sum_{\omega \in E} \Pr[\omega] \).
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Event: Example

Ω = \{\text{Red}, \text{Green}, \text{Yellow}, \text{Blue}\}

\text{Pr}\left[\text{Red}\right] = \frac{3}{10}, \quad \text{Pr}\left[\text{Green}\right] = \frac{4}{10}, \quad \text{etc.}

E = \{\text{Red}, \text{Green}\} \Rightarrow \text{Pr}\left[E\right] = \frac{3}{10} + \frac{4}{10} = \frac{3}{10} + \frac{4}{10} = \text{Pr}\left[\text{Red}\right] + \text{Pr}\left[\text{Green}\right].
Event: Example

Physical experiment

Probability model

Ω = \{Red, Green, Yellow, Blue\}

Pr[Red] = 3/10, Pr[Green] = 4/10, etc.

E = \{Red, Green\} \Rightarrow Pr[E] = \frac{3}{10} + \frac{4}{10} = \frac{3}{10} + \frac{4}{10} = Pr[Red] + Pr[Green].
Event: Example

\[ \Omega = \{ \text{Red, Green, Yellow, Blue} \} \]
Event: Example

Physical experiment

\( \Omega = \{ \text{Red, Green, Yellow, Blue} \} \)

\( Pr[\text{Red}] = \)

Probability model

- Red: 3/10
- Green: 4/10
- Yellow: 2/10
- Blue: 1/10
Event: Example

\[ \Omega = \{ \text{Red, Green, Yellow, Blue} \} \]

\[ Pr[\text{Red}] = \frac{3}{10}, \]

\[ Pr[\text{Green}] = \frac{4}{10}, \]

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Probability of exactly one heads in two coin flips?

Sample Space, \( \Omega = \{HH, HT, TH, TT\} \).

Uniform probability space: \( \text{Pr}[HH] = \text{Pr}[HT] = \text{Pr}[TH] = \text{Pr}[TT] = \frac{1}{4} \).

Event, \( E \), "exactly one heads": \( \{TH, HT\} \).

\[ \text{Pr}[E] = \sum_{\omega \in E} \text{Pr}[\omega] = \frac{|E|}{|\Omega|} = \frac{2}{4} = \frac{1}{2}. \]
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Example: 20 coin tosses.

20 coin tosses

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What is more likely?
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Answer:
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$|E_2| = \binom{20}{10} =$
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What is more likely?

$\omega_1 := (1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1), \text{ or }$

$\omega_2 := (1,0,1,1,0,0,0,1,0,1,0,1,1,0,1,1,0,0,0)$?

Answer: Both are equally likely: $Pr[\omega_1] = Pr[\omega_2] = \frac{1}{|\Omega|}$.

What is more likely?

$(E_1)$ Twenty Hs out of twenty, or
$(E_2)$ Ten Hs out of twenty?

Answer: Ten Hs out of twenty.

Why? There are many sequences of 20 tosses with ten Hs; only one with twenty Hs. $Pr[E_1] = \frac{1}{|\Omega|} \ll Pr[E_2] = \frac{|E_2|}{|\Omega|}$.

$|E_2| = \binom{20}{10} = 184,756.$
Probability of \( n \) heads in 100 coin tosses.
Probability of \( n \) heads in 100 coin tosses.

\[ \Omega = \{H, T\}^{100}; \]
Probability of $n$ heads in 100 coin tosses.

\[ \Omega = \{H, T\}^{100}; \quad |\Omega| = 2^{100}. \]
Probability of $n$ heads in 100 coin tosses.

$\Omega = \{H, T\}^{100}; \ |\Omega| = 2^{100}.$
Probability of $n$ heads in 100 coin tosses.

\[ \Omega = \{H, T\}^{100}; \; |\Omega| = 2^{100}. \]

Event $E_n = \text{‘}n\text{ heads’};$
Probability of $n$ heads in 100 coin tosses.

$\Omega = \{H, T\}^{100}; \ |\Omega| = 2^{100}.$

Event $E_n = \text{‘}n\text{ heads’}; \ |E_n| =$
Probability of $n$ heads in 100 coin tosses.

$\Omega = \{H, T\}^{100}; \ |\Omega\ | = 2^{100}.$

Event $E_n = \text{‘}n\text{ heads’}; \ |E_n| = \binom{100}{n}$
Probability of $n$ heads in 100 coin tosses.

$\Omega = \{H, T\}^{100}; \ |\Omega| = 2^{100}.$

Event $E_n = \text{`n heads'}; \ |E_n| = \binom{100}{n}$

$p_n := Pr[E_n] = \ldots$
Probability of \( n \) heads in 100 coin tosses.

\[ \Omega = \{H, T\}^{100}; \quad |\Omega| = 2^{100}. \]

Event \( E_n = \text{‘}n \text{ heads’}; \quad |E_n| = \binom{100}{n} \]

\[ p_n := Pr[E_n] = \frac{|E_n|}{|\Omega|} = \]
Probability of $n$ heads in 100 coin tosses.

$\Omega = \{H, T\}^{100}; \ |\Omega| = 2^{100}$.

Event $E_n = \text{‘n heads’}; \ |E_n| = \binom{100}{n}$

$p_n := Pr[E_n] = \frac{|E_n|}{|\Omega|} = \frac{\binom{100}{n}}{2^{100}}$
Probability of $n$ heads in 100 coin tosses.

$$\Omega = \{H, T\}^{100}; \ |\Omega| = 2^{100}. $$

Event $E_n = \text{‘n heads’}; \ |E_n| = \binom{100}{n}$

$$p_n := Pr[E_n] = \frac{|E_n|}{|\Omega|} = \frac{\binom{100}{n}}{2^{100}}$$

Observe:
Probability of $n$ heads in 100 coin tosses.

$\Omega = \{H, T\}^{100}$; $|\Omega| = 2^{100}$.

Event $E_n = \text{‘}n\text{ heads’}$; $|E_n| = \binom{100}{n}$

$p_n := Pr[E_n] = \frac{|E_n|}{|\Omega|} = \frac{\binom{100}{n}}{2^{100}}$

Observe:

- Concentration around mean:
Probability of $n$ heads in 100 coin tosses.

$\Omega = \{H, T\}^{100}; \ |\Omega| = 2^{100}.$

Event $E_n = \text{‘}n\text{ heads’}; \ |E_n| = \binom{100}{n}$

$p_n := Pr[E_n] = \frac{|E_n|}{|\Omega|} = \frac{\binom{100}{n}}{2^{100}}$

Observe:

- Concentration around mean: Law of Large Numbers;
Probability of $n$ heads in 100 coin tosses.

$\Omega = \{H, T\}^{100};$ $|\Omega| = 2^{100}.$

Event $E_n = \text{‘}n\text{ heads’};$ $|E_n| = \binom{100}{n}$

$p_n := Pr[E_n] = \frac{|E_n|}{|\Omega|} = \frac{\binom{100}{n}}{2^{100}}$

Observe:

- Concentration around mean: Law of Large Numbers;
- Bell-shape:
Probability of $n$ heads in 100 coin tosses.

$$\Omega = \{H, T\}^{100}; \ |\Omega| = 2^{100}. $$

Event $E_n = \text{‘}n\text{’ heads}; \ |E_n| = \binom{100}{n}$

$$p_n := Pr[E_n] = \frac{|E_n|}{|\Omega|} = \frac{\binom{100}{n}}{2^{100}}$$

Observe:

- Concentration around mean: Law of Large Numbers;
- Bell-shape: Central Limit Theorem.
Roll a red and a blue die.
Roll a red and a blue die.

$Pr[\text{Sum to 7}] = \frac{6}{36}$  \hspace{1cm}  $Pr[\text{Sum to 10}] = \frac{3}{36}$
Exactly 50 heads in 100 coin tosses.

Sample space: $\Omega = \text{set of 100 coin tosses}$
Exactly 50 heads in 100 coin tosses.

Sample space: $\Omega = \text{set of 100 coin tosses} = \{H, T\}^{100}$. 
Exactly 50 heads in 100 coin tosses.

Sample space: $\Omega =$ set of 100 coin tosses $= \{H, T\}^{100}$.  
$|\Omega| = 2 \times 2 \times \cdots \times 2$
Exactly 50 heads in 100 coin tosses.

Sample space: $\Omega = \text{set of 100 coin tosses} = \{H, T\}^{100}$. $|\Omega| = 2 \times 2 \times \cdots \times 2 = 2^{100}$. 
Exactly 50 heads in 100 coin tosses.

Sample space: \( \Omega = \text{set of 100 coin tosses} = \{H, T\}^{100} \).

\(|\Omega| = 2 \times 2 \times \cdots \times 2 = 2^{100} \).

Uniform probability space: \( Pr[\omega] = \frac{1}{2^{100}} \).
Exactly 50 heads in 100 coin tosses.

Sample space: $\Omega =$ set of 100 coin tosses $= \{H, T\}^{100}$. 
$|\Omega| = 2 \times 2 \times \cdots \times 2 = 2^{100}$.

Uniform probability space: $Pr[\omega] = \frac{1}{2^{100}}$.

Event $E =$ “100 coin tosses with exactly 50 heads”
Exactly 50 heads in 100 coin tosses.

Sample space: $\Omega$ = set of 100 coin tosses $= \{H, T\}^{100}$. 
$|\Omega| = 2 \times 2 \times \cdots \times 2 = 2^{100}$.

Uniform probability space: $Pr[\omega] = \frac{1}{2^{100}}$.

Event $E$ = “100 coin tosses with exactly 50 heads”

$|E|$?
Choose 50 positions out of 100 to be heads.
Exactly 50 heads in 100 coin tosses.

Sample space: \( \Omega = \) set of 100 coin tosses \( = \{H, T\}^{100} \).
\(|\Omega| = 2 \times 2 \times \cdots \times 2 = 2^{100} \).

Uniform probability space: \( Pr[\omega] = \frac{1}{2^{100}} \).

Event \( E = \) “100 coin tosses with exactly 50 heads”

\(|E|\)?

Choose 50 positions out of 100 to be heads.
\(|E| = \binom{100}{50} \).
Exactly 50 heads in 100 coin tosses.

Sample space: \( \Omega = \text{set of 100 coin tosses} = \{H, T\}^{100} \).
\(|\Omega| = 2 \times 2 \times \cdots \times 2 = 2^{100} \).

Uniform probability space: \( Pr[\omega] = \frac{1}{2^{100}} \).

Event \( E = \text{“100 coin tosses with exactly 50 heads”} \)

\(|E| ?

Choose 50 positions out of 100 to be heads.

\(|E| = \binom{100}{50} \).

\[ Pr[E] = \frac{\binom{100}{50}}{2^{100}}. \]
Calculation.
Stirling formula (for large $n$):

$$n! \approx \sqrt{2\pi n} \left(\frac{n}{e}\right)^n.$$
Calculation.

Stirling formula (for large $n$):

$$n! \approx \sqrt{2\pi n} \left( \frac{n}{e} \right)^n.$$

$$\binom{2n}{n} \approx \frac{\sqrt{4\pi n (2n/e)^{2n}}}{[\sqrt{2\pi n (n/e)^n}]^2}.$$
Calculation.
Stirling formula (for large \( n \)):

\[
\begin{align*}
n! & \approx \sqrt{2\pi n \left( \frac{n}{e} \right)^n}.
\end{align*}
\]

\[
\begin{align*}
\binom{2n}{n} & \approx \frac{\sqrt{4\pi n (2n/e)^{2n}}}{\left[ \sqrt{2\pi n (n/e)^n} \right]^2} \approx \frac{4^n}{\sqrt{\pi n}}.
\end{align*}
\]
Calculation.

Stirling formula (for large $n$):

$$n! \approx \sqrt{2\pi n} \left(\frac{n}{e}\right)^n.$$  

$$\binom{2n}{n} \approx \frac{\sqrt{4\pi n}(2n/e)^{2n}}{[\sqrt{2\pi n}(n/e)^n]^2} \approx \frac{4^n}{\sqrt{\pi n}}.$$

$$Pr[E] = \frac{|E|}{|\Omega|} =$$
**Calculation.**

**Stirling formula (for large \( n \)):**

\[
n! \approx \sqrt{2\pi n} \left(\frac{n}{e}\right)^n.
\]

\[
\binom{2n}{n} \approx \frac{\sqrt{4\pi n(2n/e)^{2n}}}{\sqrt{2\pi n(n/e)^n}} \approx \frac{4^n}{\sqrt{\pi n}}.
\]

\[
Pr[E] = \frac{|E|}{|\Omega|} = \frac{|E|}{2^{2n}} = \ldots
\]
Calculation.
Stirling formula (for large $n$):

$$n! \approx \sqrt{2\pi n} \left( \frac{n}{e} \right)^n.$$

$$\binom{2n}{n} \approx \frac{\sqrt{4\pi n}(2n/e)^{2n}}{[\sqrt{2\pi n(n/e)^n}]^2} \approx \frac{4^n}{\sqrt{\pi n}}.$$

$$Pr[E] = \frac{|E|}{|\Omega|} = \frac{|E|}{2^{2n}} = \frac{1}{\sqrt{\pi n}} = \ldots$$
Calculation.

Stirling formula (for large $n$):

$$n! \approx \sqrt{2\pi n} \left( \frac{n}{e} \right)^n.$$  

$$\binom{2n}{n} \approx \frac{\sqrt{4\pi n (2n/e)^{2n}}}{[\sqrt{2\pi n (n/e)^n}]^2} \approx \frac{4^n}{\sqrt{\pi n}}.$$  

$$Pr[E] = \frac{|E|}{|\Omega|} = \frac{|E|}{2^{2n}} = \frac{1}{\sqrt{\pi n}} = \frac{1}{\sqrt{50\pi}} \approx .08.$$
Exactly 50 heads in 100 coin tosses.

\[ \Pr[n \text{Hs out of } 2n] = \frac{\binom{2n}{n}}{2^{2n}} \]
Lecture 13: Summary

1. Random Experiment
2. Probability Space: $\Omega; Pr[\omega] \in [0, 1]; \sum_\omega Pr[\omega] = 1.$
3. Uniform Probability Space: $Pr[\omega] = 1/|\Omega|$ for all $\omega \in \Omega$.
4. Events: subsets of $\Omega$.
   \[ Pr[E] = \sum_{\omega \in E} Pr[\omega]. \]