CS70: Counting

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Today:

- Balls and bins.
- Sum rule.
- Combinatorial proofs.
- Maybe start review?
What we’ve learned so far

Sample $k$ items out of $n$.

<table>
<thead>
<tr>
<th></th>
<th>With Replacement</th>
<th>Without Replacement</th>
</tr>
</thead>
<tbody>
<tr>
<td>Order matters</td>
<td>$n^k$</td>
<td>$\frac{n!}{(n-k)!}$</td>
</tr>
<tr>
<td>Order doesn’t matter</td>
<td>$\binom{n+k-1}{n-1}$</td>
<td>$\binom{n}{k}$</td>
</tr>
</tbody>
</table>
A unifying example

Hats! Say I have 10 different hats. I’m thinking of how many different outfits I have for one week:

▶ Case 1: After I wear a hat I can wear it again. (Sampling with replacement)
  ▶ Subcase: I care about which day I wore what: blue hat on Monday is different than blue hat on Tuesday. duh. (Order matters.)
    10 options for Monday, 10 options for Tuesday... $10^7$.
  ▶ Subcase: I don’t care about which day I wore what, I just care which hats I ending up wearing. (Order doesn’t matter.)
    How many samples? a day is a sample, so 7. From how big of a set? 10 hats.
    $$\binom{n+k-1}{n-1} = \binom{10+7-1}{10-1} = \binom{16}{9} = 11440$$
Case 2: After I wear a hat I destroy it. (Sampling without replacement)

- Subcase: I care about which day I wore what: blue hat on Monday is different than blue hat on Tuesday. (Order matters.)
  10 options for Monday, 9 options for Tuesday...
  \[ \frac{n!}{(n-k)!} = \frac{10!}{3!} = 604800. \]

- Subcase: I don’t care about which day I wore what. (Order doesn’t matter.)
  \[ \binom{10}{7} = 120 \]
How many (non-negative) solutions to \( x + y = 10 \)?
Easy: \( x = 0, y = 10, x = 1, y = 9, x = 2, y = 8, \ldots, x = 10, y = 0 \). So 11 solutions.
Same as 10 stars, and 1 bar.
\( x = 3, y = 7 \) : ⋆ ⋆ ⋆ | ⋆ ⋆ ⋆ ⋆ ⋆ ⋆ ⋆ ⋆ ⋆ ⋆ 
Think of a star as the number 1.

How many ways to make an 8 problem midterm such the total points add up to 100?
100 stars, 7 bars.
Balls in bins.

"$k$ Balls in $n$ bins" $\equiv$ "$k$ samples from $n$ possibilities."

"indistinguishable balls" $\equiv$ "order doesn’t matter"

"only one ball in each bin" $\equiv$ "without replacement"
Balls and bins

How many 5 digit numbers?
Throwing 5 numbered balls in 10 (numbered) bins:

Picture has number 62280.
5 samples from 10 possibilities with replacement (order matters): $10^5$
Balls and bins

How many 5 digit numbers without repeating a digit?
Throwing 5 numbered balls in 10 (numbered) bins, at most one ball per bin:

Picture has number 62480.
5 samples from 10 possibilities without replacement (order matters):
\[ \frac{10!}{5!} \]
How many 3 card poker hands?
Throwing 3 indistinguishable balls in 52 (numbered) bins. At most 1 ball per bin:

Picture has cards 1, 3 and 52.
3 samples from 52 possibilities without replacement (order doesn’t matter): \( \binom{52}{3} \)
Dividing 5 dollars among Alice, Bob and Eve.
5 indistinguishable balls into 3 (numbered) bins:

Picture: Alice 3, Bob 0, Eve 2.
5 samples from 3 possibilities with replacement (order doesn’t matter): $\binom{7}{2}$
Sum Rule

Two indistinguishable jokers in 54 card deck. How many 5 card poker hands? 

**Sum rule: Can sum over disjoint sets.**

No jokers or exactly one joker or exactly two jokers

\[
\binom{52}{5} + \binom{52}{4} + \binom{52}{3}.
\]

Two distinguishable jokers in 54 card deck. How many 5 card poker hands (distinguishable jokers)?

No jokers or exactly one of two jokers or exactly two jokers

\[
\binom{52}{5} + 2 \times \binom{52}{4} + \binom{52}{3}
\]

Wait a minute! Same as choosing 5 cards from 54 or

\[
\binom{54}{5}
\]

**Theorem:** \(\binom{54}{5} = \binom{52}{5} + 2 \times \binom{52}{4} + \binom{52}{3}\).

**Proof:** Above is combinatorial proof.
Algebraic proof

\[
\binom{54}{5} = \binom{52}{5} + 2 \times \binom{52}{4} + \binom{52}{3}.
\]

Proof:

\[
\binom{54}{5} = \frac{54!}{5!49!}, \quad \binom{52}{5} = \frac{52!}{5!47!}, \quad \binom{52}{4} = \frac{52!}{4!48!}, \quad \binom{52}{3} = \frac{52!}{3!49!}
\]

\[
RHS = \frac{52!}{5!47!} + 2 \times \frac{52!}{4!48!} + \frac{52!}{3!49!} = \frac{52!(4!48!3!49!+2\times5!47!3!49!+5!47!4!48!)}{5!47!4!48!3!49!} = \frac{54!}{5!49!}
\]

49! and 5! cancel out. Cross multiply and get:

\[
54!47!4!48!3! = 52!(4!48!3!49!+2\times5!47!3!49!+5!47!4!48!)
\]

I tried this for a while.......
Combinatorial Proofs.

**Theorem:** \( \binom{n}{k} = \binom{n}{n-k} \)

**Proof:** How many subsets of size \( k \)? \( \binom{n}{k} \)

How many subsets of size \( k \)?
Choose a subset of size \( n-k \)
    and what’s left out is a subset of size \( k \).
Choosing a subset of size \( k \) is same
    as choosing \( n-k \) elements to not take.
\( \implies \binom{n}{n-k} \) subsets of size \( k \).
Pascal’s Triangle

Row \( n \): coefficients of \((1 + x)^n = (1 + x)(1 + x) \cdots (1 + x)\).

Zero-th row: \((1 + x)^0 = 1\)

First row: \((1 + x)^1 = x + 1\). Coefficients: 1 and 1

Second row: \((1 + x)^2 = 1 + 2x + x^2\). Coefficients: 1, 2 and 1

Third row: \((1 + x)^3 = 1 + 3x + 3x^2 + x^3\). Coefficients: 1, 3, 3 and 1

.....

Foil?? I hate this already...
Pascal’s Triangle

```
 1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
```

Simplify: collect all terms corresponding to $x^k$.

Coefficient of $x^k$ is $\binom{n}{k}$: choose $k$ factors where $x$ is in product.

$(1 + x)(1 + x)(1 + x)(1 + x)$: Coefficients of $x^2$ come from: first and second term, first and third term, first and fourth, second and third, second and fourth, third and fourth. 6 of them, aka $\binom{4}{2}$.

\[
\begin{align*}
\binom{0}{0} & \quad \binom{1}{0} \quad \binom{1}{1} \\
\binom{2}{0} & \quad \binom{2}{1} \quad \binom{2}{2} \\
\binom{3}{0} & \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3}
\end{align*}
\]

Pascal’s rule $\implies \binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$. 
Combinatorial Proofs.

**Theorem:** \( \binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1} \).

**Proof:** How many size \( k \) subsets of \( n+1 \)? \( \binom{n+1}{k} \).

How many size \( k \) subsets of \( n+1 \)?
The ones that contain the first element plus the ones that don’t contain the first element.

How many contain the first element?
Pick the first. Then I need to choose \( k-1 \) more from remaining \( n \) elements.
\[ \implies \binom{n}{k-1} \]

How many don’t contain the first element?
Need to choose \( k \) elements from remaining \( n \) elements.
\[ \implies \binom{n}{k} \]

So, \( \binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k} \). □
Combinatorial Proof.

**Theorem:** \( \binom{n}{k} = \binom{n-1}{k-1} + \cdots + \binom{k-1}{k-1} \).

**Proof:**

Left Hand Side (LHS): Size \( k \) subsets of \( n \).

Consider size \( k \) subset where \( i \) is the smallest element chosen.

\[
\{1, \ldots, i, \ldots, n\}
\]

Must choose \( k - 1 \) elements from \( n - i \) remaining elements.

\[ \implies \binom{n-i}{k-1} \text{ such subsets.} \]

1 is smallest element chosen: \( \binom{n-1}{k-1} \) choices for the rest.

2 is smallest element chosen: \( \binom{n-2}{k-1} \) choices for the rest.

and so on.

Add them up to get the total number of subsets of size \( k \) which is also \( \binom{n}{k} \).
Theorem: \(2^n = \binom{n}{n} + \binom{n}{n-1} + \cdots + \binom{n}{0}\)

Proof: How many subsets of \(\{1, \ldots, n\}\)?
Construct a subset with sequence of \(n\) choices:
- element \(i\) is in or is not in the subset: 2 possibilities.
First rule of counting: \(2 \times 2 \cdots \times 2 = 2^n\) subsets.

How many subsets of \(\{1, \ldots, n\}\)?
\(\binom{n}{i}\) = subsets of size \(i\).
A subset has size either 0, or 1, or 2, \ldots, or \(n\)
Sum over \(i\) to get total number of subsets.
Simple Inclusion/Exclusion

**Sum Rule:** For disjoint sets $S$ and $T$, $|S \cup T| = |S| + |T|$

Used to reason about all subsets by adding number of subsets of size 0, 1, 2, 3, …

Also reasoned about subsets that contained or didn’t contain an element. (E.g., first element, first $i$ elements.)

**Inclusion/Exclusion Rule:** For any $S$ and $T$,

$$|S \cup T| = |S| + |T| - |S \cap T|.$$  

**Example:** How many 10-digit phone numbers have 7 as their first or second digit?

$S =$ phone numbers with 7 as first digit. $|S| = 10^9$

$T =$ phone numbers with 7 as second digit. $|T| = 10^9$.

$S \cap T =$ phone numbers with 7 as first and second digit. $|S \cap T| = 10^8$.

Answer: $|S| + |T| - |S \cap T| = 10^9 + 10^9 - 10^8$. 
Summary.

Inclusion/Exclusion: two sets of objects.
   Add number of each subtract intersection of sets.
Sum Rule: If disjoint just add.

Combinatorial Proofs: Identity from counting same in two ways.
Pascal’s Triangle Example: \( \binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k} \).
   RHS: Number of subsets of \( n+1 \) items size \( k \).
   LHS: \( \binom{n}{k-1} \) counts subsets of \( n+1 \) items with first item.
   \( \binom{n}{k} \) counts subsets of \( n+1 \) items without first item.
Disjoint – so add!