1. Logic

Decide whether each of the following is true or false and justify your answer:

a) \( \forall x (P(x) \wedge Q(x)) \equiv \forall x P(x) \wedge \forall x Q(x) \)

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False. If \( P(1) \) is true, \( Q(1) \) is false, \( P(2) \) is false and \( Q(2) \) is true, the left-hand side will be true, but the right-hand side will be false.

b) \( \forall x (P(x) \lor Q(x)) \equiv \forall x P(x) \lor \forall x Q(x) \)

2. (Proof)

A perfect square is an integer \( n \) of the form \( n = m^2 \) for some integer \( m \). Prove that every odd perfect square is of the form \( 8k + 1 \) for some integer \( k \).

Let \( n = m^2 \) for some integer \( m \). Since \( n \) is odd, \( m \) is also odd, i.e., of the form \( m = 2l + 1 \) for some integer \( l \). Then, \( m^2 = 4l^2 + 4l + 1 = 4(l + 1) + 1 \). Since one of \( l \) and \( l + 1 \) must be even, \( l(l + 1) \) is of the form \( 2k \) and \( n = m^2 = 8k + 1 \).

3. Contradiction

Prove that \( 2^{1/n} \) is not rational for any integer \( n > 3 \). [Hint : Fermat’s Last Theorem and the method of contradiction]

If not, then there exists an integer \( n > 3 \) such that \( 2^{1/n} = \frac{p}{q} \) where \( p, q \) are positive integers. Thus, \( 2q^n = p^n \), and this implies,

\[ q^n + q^n = p^n \]

which is a contradiction to the Fermat’s Last Theorem.

4. Problem solving

Prove that if you put \( n + 1 \) apples into \( n \) boxes, any way you like, then at least one box must contain at least 2 apples. This is known as the pigeonhole principle.

Suppose this is not the case. Then all the boxes would contain at most 1 apple. Then the maximum number of apples we could have would be \( n \), but this is a contradiction since we have \( n + 1 \) apples.

5. Numbers of Friends

Prove that if there are \( n \geq 2 \) people at a party, then at least 2 of them have the same number of friends at the party. Answer: Suppose the contrary that everyone has a different number of friends at the party.
Since the number of friends that each person can have ranges from 0 to $n - 1$, we conclude that for every $i \in \{0, 1, \ldots, n - 1\}$, there is exactly one person who has exactly $i$ friends at the party. In particular, there is one person who has $n - 1$ friends (i.e., friends with everyone), and there is one person who has 0 friends (i.e., friends with no one), which is a contradiction.