

**1. Coins**

Flip a biased coin until you get a heads. Let  $p$  be the chance of heads.

- (a) What's the chance that there are  $k$  heads?
  
  
  
  
- (b) What's the chance that there are  $k$  tails?
  
  
  
  
- (c) Find the expected number of heads, and expected number of tails. (Hint: Find a recurrence relation.)
  
  
  
  
- (d) What's the expected number of flips?
  
  
  
  
- (e) You know  $p \geq 25\%$ . How many iterations of this game do you need to find  $p$  to within 10% with a confidence of 90%? The variance of a geometric distribution is  $\frac{1-p}{p^2}$ .

**2. Law of Large Numbers**

We wish to find the average temperature in the United States. We divide up the country into a grid, each section with area  $1 \text{ mi}^2$ . First, we randomly sample a section and measure the temperature. We then repeat the sampling process (with replacement) and average our results. Prove that the average sampled temperature approaches the average temperature as the number of measurements approaches infinity. Be rigorous.

### 3. Boltzmann

The Boltzmann distribution is a very important distribution in statistical mechanics. (In mathematics, this is also known as the Gibbs measure.) There are a finite number of states  $i = 1, 2, \dots, n$ , each equipped with an energy  $E_i$ . A heat bath is applied and the particles settle to thermal equilibrium (for this problem, 60°F, which corresponds to a fundamental temperature  $\tau = 4 \times 10^{-21}$  Joules). The ratio of probability between two states  $i, j$  is determined by  $e^{\frac{E_j - E_i}{\tau}}$ . If there are  $N_i$  particles in each state, this probability ratio is also the ratio of the number of particles  $\frac{N_i}{N_j}$ .

- (a) Suppose a system has 100 identical particles and two states, with energies  $E_1 = 8 \times 10^{-21}$  Joules and  $E_2 = 12 \times 10^{-21}$  Joules. How many ways can you distribute the particles? What's the number of particles in each state?
- (b) Another system has 150 particles and three states, with energies  $E_1 = 8 \times 10^{-21}$  Joules,  $E_2 = 12 \times 10^{-21}$  Joules, and an unknown  $E_3$ . There are 10 particles in state 2. How many particles are in states 1 and 3? What's the energy of state 3? Use  $e \approx 2.7$ .
- (c) If there are  $n$  states (each state  $i$  with energy  $E_i$ ), what's the probability of being in each state?

### 4. Coupon Collector Revisited

There are  $n$  baseball cards Jonny is trying to collect! Each day, Jonny buys a cereal box and dumps out all the cereal, searching for the baseball card. Each box contains each of the  $n$  cards with equal probability.

- (a) What's the expected number of days until Jonny gets one unique card? Two unique cards?
- (b) Develop a formula for the expected number of days until Jonny gets  $k$  unique cards. On average, how many days will it take Jonny to collect all the baseball cards? Approximate your sum with an integral.
- (c) Jonny and his sister, Jill, are working together to collect the cards. Each day, they both buy cereal boxes. On average, how many days will it take to collect  $k$  unique baseball cards? How many days until they collect them all? Be exact.