CS 70 Discrete Mathematics and Probability Theory
Summer 2015 Chung-Wei Lin Discussion 2A

The Traditional Propose & Reject Algorithm

- **Every Morning**: Each man proposes to the most preferred woman on his list who has not yet rejected him.
- **Every Afternoon**: Each woman collects all the proposals she received in the morning; to the man she likes best, she responds "maybe, come back tomorrow" (she now has him *on a string*), and to the others, she says "never".
- **Every Evening**: Each rejected man crosses off the woman who rejected him from his list.

The above loop is repeated each successive day until each woman has a man on a string; on this day, each woman marries the man she has on a string.

1. Stable Marriage

Consider the following preference lists for 4 men ($1,2,3$ and 4) and 4 women (A,B,C and D):

Men	Preferences
1	$A > D > C > B$
2	$B > D > A > C$
3	$D > A > B > C$
4	$D > A > B > C$

Women	Preferences
A	$1 > 3 > 4 > 2$
B	$3 > 1 > 4 > 2$
C	$2 > 1 > 3 > 4$
D	$4 > 3 > 2 > 1$

(a) Run the Traditional Propose & Reject Algorithm with the preference lists above and list the final stable couples created.

(b) Now, assume that 1 and A are at the top of all the preference lists for the women and men respectively. What can you say about them in every stable pairing?

(c) Now, assume that 4 and D are at the bottom of all the preference lists for the women and men respectively. What can you say about them in every stable pairing?

Answer:

(a) $(1,A),(2,C),(3,B),(4,D)$.

(b) Always together. 1 proposes to A on the first day, and she always has him on a string.

(c) Always together. If not, then there will be rogue couples.

2. Universal Preferences

Suppose that preferences are universal, that is all n men share the preferences $W_1 > W_2 > \ldots > W_n$ and all women share the preferences $M_1 > M_2 > \ldots > M_n$.

(a) What result do we get from running the algorithm with men proposing?

(b) What result do we get from running the algorithm with women proposing?

(c) What does this tell us about the number of stable matchings?

Answer:

(a) For all i, M_i is matched with W_i.

(b) For all i, M_i is matched with W_i.

(c) Since the male-optimal and female-optimal matchings are the same, this matching is the unique stable matching.

3. Proofs about the Traditional Propose & Reject Algorithm

Prove the following statements about the Traditional Propose & Reject Algorithm:

(a) In any execution of the algorithm, if a woman receives a proposal in day i, she receives some proposal on every day thereafter until termination.

(b) In any execution of the algorithm that takes k days, there must be some woman who does not receive a proposal in day $k-1$.

(c) In any execution of the algorithm, if woman W receives no proposal in day i, then she receives no proposal in any previous day $j, 1 \leq j < i$.

(d) From the above parts, can you prove that in any execution of the algorithm, there is at least one woman who only receives a single proposal?

Answer:

(a) If a woman receives a proposal on day i, then, by the Improvement Lemma, she will always have someone as good or better proposing to her every day after day i.

(b) Proof by contradiction: assume that every woman receives a proposal on or before day $k-1$. By part (a), she will continue to receive proposals on day $k-1$. But since there are n men and n women, this means that every woman is proposed to by a unique man (pigeonhole principle). If that is the case, the algorithm should already have terminated on this day. Contradiction.

(c) Proof by contraposition: if she does receive a proposal on some day $j \geq 1$, then she will receive a proposal on day $i > j$ from part(a).

(d) From parts (b) and (c), we see that there is at least one woman who will receive a proposal on the last day of the running of the algorithm and will have received no proposal before this date. This also means that she will only receive only one proposal.