CS 70 Discrete Mathematics and Probability Theory
Summer 2015 Chung-Wei Lin HW 4

Due ~~Monday July 20~~ Tuesday July 21 at Noon

Please write your answer of each question on a separated page.

1. **Self-Grading Shuffle** (12 points, 3 points for each part)

   A TA likes the students to perform self-grading on their homework in section. 20 students walk in and sit down with their homework in front of them. At the beginning of class, the TA walk around and randomly shuffle the homework. For Parts (b)–(d), assume there is no tie in age.

   (a) What is the total number of possible (student and homework) configurations the TA could have after shuffling?

   (b) How many configurations are there where the oldest student gets his/her own homework back?

   (c) How many configurations are there where at least 1 of the 2 oldest students get their own homework back?

   (d) How many configurations are there where at least 1 of the 3 oldest students get their own homework back?

2. **Combinatorial Proof** (30 points, 10 points for each part)

   (a) n males and n females apply for the CS major at UC Berkeley. The CS department only has n seats available. How many ways can it admit students? Use the above story for a combinatorial argument to prove the following identity:

   $$\binom{2n}{n} = \binom{n}{0}^2 + \binom{n}{1}^2 + \cdots + \binom{n}{n}^2.$$

   (b) Following Part (a), the department decides to provide two fellowships — one for a male student and the other one for a female student. Use the above story for a combinatorial argument to prove the following identity:

   $$\sum_{k=1}^{n-1} k \cdot (n-k) \cdot \binom{n}{k}^2 = n^2 \cdot \binom{2n-2}{n-2}.$$

   (c) Now, come up with your own story for a combinatorial argument to prove the following identity.

   $$n2^{n-1} = \binom{n}{1} + 2\binom{n}{2} + \cdots + (n-1)\binom{n}{n-1} + n\binom{n}{n}$$

3. **Monty Hall Problem** (24 points, 6/6/3/3/3/3 points for each part)

   (a) Explain the Monty Hall problem in your own words. Pretend you are teaching yourself as you were at the beginning of this term.

   (b) Define the outcome of the game (up to the point where the contestant makes the final decision) by a triple of the form (i, j, k), where $i, j, k \in \{1, 2, 3\}$. The values i, j, k respectively specify the location of the prize, the initial door chosen by the contestant, and the door opened by Carol. What is the sample space of this problem? Draw it as a tree structure.

   (c) What is the probability of each sample point?

   (d) What is the event of interest?

   (e) If the contestant is using the "sticking strategy," which sample points are in this event? What is the probability of this event?

   (f) If the contestant is using the "switching strategy," which sample points are in this event? What is the probability of this event?

4. **Profound Experimentation** (18 points, 3 points for each part)

   You perform an exhilarating and groundbreaking experiment upon the outcome of flipping an unbiased coin 5 times (order matters).

   (a) How many total outcomes are there?

   (b) What is the probability that there is exactly 1 Head?

   (c) What is the probability that there is at least 1 Head?

   (d) What is the probability that there are more Heads than Tails?

   (e) What is the probability that at least one of the first two flips is a Tail?

   (f) What is the probability that there are exactly 4 Heads in a row?

5. **Wedding Chaos** (9 points, 3 points for each part)

   You have been invited to your cousin's wedding celebration. She has ordered special suits and dresses, custom fitted, for you and her favorite friends from college. All together, there are 8 men and 8 women. Each man has a custom suit and each woman has a custom dress. Unfortunately, when you arrive you realize the tailor has forgotten to label which belongs to whom.

   (a) The event starts soon, and you are all running out of time. The men all decide to randomly grab a suit and put it on. What is the probability that all the men are wearing the right suit?

   (b) The women have a better plan. By comparing the dresses, they first separate them into two piles; one for the smaller four dresses, the other for the larger four dresses. Then, the smaller four women choose a dress from the small pile and the larger from the large pile. What is the chance that the women will all end up wearing the right dress?

   (c) Everyone is finally dressed, but now a dance is starting. Your cousin has told you that she expects four couples to dance, but hasn't said who. Four of the men and four of the women randomly agree to dance. It turns out she really only wanted people who don't wear glasses dancing (she isn't your kindest cousin), because she thinks glasses would look bad in her wedding photographs. If one of the women wears glasses, and two of the men wear glasses, how likely is it that you didn't ruin her wedding photos?

6. **Random Team** (9 points, 3 points for each part)

   There is a set of 100 people, all *strictly ordered* in some way (the first person is the best). You randomly pick a team of 50.

   (a) What is the probability that at least one member of your team will be better than the median?

   (b) What is the probability that at least one member of your team will be better than the k-th person for $1 \leq k \leq 51$?

   (c) What is the probability that at least one member of your team will be better than the k-th person for $52 \leq k \leq 100$?.

7. **Roll the Caltrop** (18 points, 3 points for each part)

   You are playing Dungeons and Dragons with your friends. As usual, you spend far too long designing your character, but this time you have a good reason: you are thinking about the probabilities involved in rolling a 4 sided die (sometimes called a caltrop) three times.

   (a) How large is the sample space of rolling the caltrop three times?

   (b) How likely is it that you will roll the same number all three times?

   (c) What is the chance that you will roll exactly two 3's?

   (d) It turns out your friend has a caltrop which he tinkered with to make it more likely to land showing a 1. It lands on 1 half of the time, and 2, 3, and 4 each have an equal probability of $\frac{1}{6}$. How large is the sample space of rolling the caltrop three times?

   (e) Following Part (d), how likely is it that you will roll the same number all three times? Is it larger than, equal to, or smaller than the answer in Part (b)?

   (f) Following Part (d), what is the chance that you will roll exactly two 3's? Is it larger than, equal to, or smaller than the answer in Part (c)?