



Long Term Fraction of Time in States

Theorem Let X_n be an irreducible Markov chain with invariant distribution π . Then, for all i, $\frac{1}{n}\sum_{m=0}^{n-1} 1\{X_m = i\} \to \pi(i)$, as $n \to \infty$.

Example 2:



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Theorem Let X_n be an irreducible Markov chain with invariant distribution π .

Then, for all *i*,

$$\frac{1}{n}\sum_{m=0}^{n-1} 1\{X_m = i\} \to \pi(i), \text{ as } n \to \infty.$$

The left-hand side is the fraction of time that $X_m = i$ during steps 0, 1, ..., n - 1. Thus, this fraction of time approaches $\pi(i)$.

Proof: Lecture note 24 gives a plausibility argument.

Convergence to Invariant Distribution

Question: Assume that the MC is irreducible. Does π_n approach the unique invariant distribution π ?

Answer: Not necessarily. Here is an example:



Assume $X_0 = 1$. Then $X_1 = 2, X_2 = 1, X_3 = 2, \dots$

Thus, if $\pi_0 = [1,0]$, $\pi_1 = [0,1]$, $\pi_2 = [1,0]$, $\pi_3 = [0,1]$, etc. Hence, π_0 does not converge to $\pi = [1/2, 1/2]$.

Notice, all cycles or closed walks have even length.

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Theorem Let X_n be an irreducible Markov chain with invariant distribution π . Then, for all i, $\frac{1}{n}\sum_{m=0}^{n-1} 1\{X_m = i\} \rightarrow \pi(i)$, as $n \rightarrow \infty$. **Example 1:**



The fraction of time in state 1 converges to 1/2, which is $\pi(1)$.

Periodicity

Definition: Periodicity is gcd of the lengths of all closed walks. Previous example: 2. Definition If periodicity is 1, Markov chain is said to be aperiodic. Otherwise, it is periodic. Example



[A]: Closed walks of length 3 and length 4 \implies periodicity = 1. [B]: All closed walks multiple of 3 \implies periodicity =2.

Convergence of π_n

Theorem Let X_n be an irreducible and aperiodic Markov chain with invariant distribution π . Then, for all $i \in \mathcal{X}$,

 $\pi_n(i) \to \pi(i)$, as $n \to \infty$.



CS70: Continuous Probability.



Convergence of π_n

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Example



Uniformly at Random in [0,1].

Choose a real number X, uniformly at random in [0,1]. What is the probability that X is exactly equal to 1/3? Well, ..., 0.



What is the probability that X is exactly equal to 0.6? Again, 0. In fact, for any $x \in [0, 1]$, one has Pr[X = x] = 0. How should we then describe 'choosing uniformly at random in [0, 1]'? Here is the way to do it:

 $Pr[X \in [a, b]] = b - a, \forall 0 \le a \le b \le 1.$

Makes sense: b - a is the fraction of [0,1] that [a, b] covers.

Summary Markov Chains: $Pr[X_{n+1} = j | X_0, ..., X_n = l] = P(i, j)$ • FSE: $\beta(i) = 1 + \sum_j P(i, j)\beta(j); \alpha(i) = \sum_j P(i, j)\alpha(j)$. • $\pi_n = \pi_0 P^n$ • π is invariant iff $\pi P = \pi$ • Irreducible \Rightarrow one and only one invariant distribution π • Irreducible \Rightarrow fraction of time in state *i* approaches $\pi(i)$ • Irreducible + Aperiodic $\Rightarrow \pi_n \to \pi$. • Calculating π : One finds $\pi = [0, 0, ..., 1]Q^{-1}$ where $Q = \cdots$.

Uniformly at Random in [0, 1].

Let [a, b] denote the **event** that the point X is in the interval [a, b].

$$Pr[[a,b]] = \frac{\text{length of } [a,b]}{\text{length of } [0,1]} = \frac{b-a}{1} = b-a.$$

Intervals like $[a,b] \subseteq \Omega = [0,1]$ are **events.** More generally, events in this space are unions of intervals. Example: the event *A* - "within 0.2 of 0 or 1" is $A = [0,0.2] \cup [0.8,1]$. Thus,

$$Pr[A] = Pr[[0, 0.2]] + Pr[[0.8, 1]] = 0.4.$$

More generally, if A_n are pairwise disjoint intervals in [0, 1], then

$$Pr[\cup_n A_n] := \sum_n Pr[A_n].$$

Many subsets of [0,1] are of this form. Thus, the probability of those sets is well defined. We call such sets events.







General Random Choice in R

Let F(x) be a nondecreasing function with $F(-\infty) = 0$ and $F(+\infty) = 1$. Define X by $Pr[X \in (a, b]] = F(b) - F(a)$ for a < b. Also, for $a_1 < b_1 < a_2 < b_2 < \cdots < b_n$,

 $\begin{aligned} \Pr[X \in (a_1, b_1] \cup (a_2, b_2] \cup (a_n, b_n]] \\ &= \Pr[X \in (a_1, b_1]] + \dots + \Pr[X \in (a_n, b_n]] \\ &= F(b_1) - F(a_1) + \dots + F(b_n) - F(a_n). \end{aligned}$

Let $f(x) = \frac{d}{dx}F(x)$. Then,

 $Pr[X \in (x, x + \varepsilon]] = F(x + \varepsilon) - F(x) \approx f(x)\varepsilon.$

Here, F(x) is called the cumulative distribution function (cdf) of X and f(x) is the probability density function (pdf) of X.

To indicate that *F* and *f* correspond to the RV *X*, we will write them $F_X(x)$ and $f_X(x)$.

Example: CDF

Example: hitting random location on gas tank. Random location on circle.



Random Variable: *Y* distance from center. Probability within *y* of center:

$$Pr[Y \le y] = \frac{\text{area of small circle}}{\text{area of dartboard}}$$
$$= \frac{\pi y^2}{\pi} = y^2.$$

Hence.

$$F_{Y}(y) = \Pr[Y \le y] = \begin{cases} 0 & \text{for } y < 0\\ y^{2} & \text{for } 0 \le y \le 1\\ 1 & \text{for } y > 1 \end{cases}$$

$\Pr[X \in (x, x + \varepsilon)]$



Thus, the pdf is the 'local probability by unit length.' It is the 'probability density.'

Calculation of event with dartboard..

Probability between .5 and .6 of center? Recall CDF.

$$F_{Y}(y) = \Pr[Y \le y] = \begin{cases} 0 & \text{for } y < 0\\ y^{2} & \text{for } 0 \le y \le 1\\ 1 & \text{for } y > 1 \end{cases}$$

 $\begin{aligned} Pr[0.5 < Y \le 0.6] &= Pr[Y \le 0.6] - Pr[Y \le 0.5] \\ &= F_Y(0.6) - F_Y(0.5) \\ &= .36 - .25 \\ &= .11 \end{aligned}$



