1. Strengthening the induction hypothesis

Suppose we have a supply of L-shaped tiles (2x2 tiles with a 1x1 square removed), and a $2^n \times 2^n$ grid ($n \geq 1$), prove that we can cover the grid with tiles such that only one square is left uncovered.

Answer: We prove a stronger version of the hypothesis, that one corner of the grid is left uncovered.

Base Case We can do this with one tile.

Inductive Hypothesis Assume that the claim is true for $2^n \times 2^n$ grids.

Inductive Step We arrange three copies of a $2^n \times 2^n$ grids with their corners facing together, and fill that space with one tile. Then place the fourth $2^n \times 2^n$ grid with the missing corner in the corner of the $2^{n+1} \times 2^{n+1}$ grid.

2. Recursion and induction

Working at the local pizza parlor, I have a stack of unbaked pizza doughs. For a most pleasing presentation, I wish to arrange them in order of size, with the largest pizza on the bottom. I know how to place my spatula under one of the pizzas and flip over the whole stack above the spatula (reversing their order). This is the only move I know that can change the order of the stack; however, I am willing to keep repeating this move until I get the stack in order. Give a recursive procedure to get the pizzas in order. Prove by induction that your procedure correctly orders the pizzas.

Answer: The recursive procedure SORT will sort the top-most k pizza doughs:

```
function SORT(k)
    if n = 1 then
        return                                          ▷ 1 pizza is already sorted
    else
        Place spatula below largest pizza amongst top k pizzas and flip
        Place spatula below k-th pizza from top and flip
        SORT(k − 1)
    end if
end function
```

We prove its correctness by induction on k.

Base Case If there is only one pizza, it is already sorted

Inductive Hypothesis Suppose SORT(k) correctly sorts the top-most k pizzas in the stack.

Inductive Step Calling SORT($k+1$) the sequence of two flips will bring the largest pizza in the top-most $k+1$ pizzas to the bottom of the stack. Then calling SORT(k) will correctly sort the remaining k pizzas. Thus the top-most $k+1$ pizzas are sorted after calling SORT($k+1$).

3. More Induction

Suppose that I start with 0 written on a piece of paper. Each minute, I choose a digit written on the paper and erase it. If it was 0, I replace it with 010. If it was 1, I replace it with 1001. Prove that no matter which digits I choose and no matter how long the process continues, I never end up with two 1's in a row.

Answer:

Base Case After the first step, we get 010, which does not have two 1's in a row.

Inductive Hypothesis After n steps, we do not have two 1's in a row.

Inductive Step At the $n+1$-st step, if we choose a 1, its neighbor(s) must be 0's, otherwise the inductive hypothesis will be violated. Thus after changing 1 to 1001, we still don't have two 1's in a row. If we choose a 0, after changing its value its neighbors would still have 0's next to them, and thus we do not introduce any pairs of 1's no matter which digit we change. Therefore, after the $n+1$-st step, there are no two 1's in a row.

4. Dividing n-gon

Assume that any simple (but not necessarily convex) n-gon ($n > 3$) has a diagonal (line between two non-adjacent vertices) that lies completely within the n-gon. Show that any such n-gon ($n \geq 3$) can be divided into $n-2$ triangles such that all vertices of each triangle are vertices of the n-gon.

Answer: We run strong induction over n:

Base Case $n = 3$. This is a triangle.

Inductive Hypothesis Assume that the claim is true for all n-gons, $n \geq 3$.

Inductive Step For a $(n+1)$-gon, a diagonal divides it into two smaller polygons. Suppose one of them is a k-gon, then the other is a $(n-k+3)$-gon, where $k \geq 3$. (The two vertices at either end of the diagonal are repeated, so there are a total of $n+3$ vertices between the two polygons.) By the inductive hypothesis, the first polygon can be divided into $k-2$ triangles, and the second into $(n-k+3)-2$ triangles. The total number of triangles is $k-2+(n-k+3)-2 = n-1$. Thus, an $(n+1)$-gon can be divided into $(n+1)-2$ triangles.

5. Fibonacci Sums

The sequence of Fibonacci numbers is defined by: $F_1 = F_2 = 1$ and $F_{n+1} = F_n + F_{n-1}$ if $n \geq 2$. Thus the sequence starts with 1, 1, 2, 3, 5, 8, 13 ... While not all natural numbers are Fibonacci numbers, interestingly, every natural number can be written as the sum of different Fibonacci numbers. Prove this result. (Hint: Use strong induction.)

Answer: We run strong induction over n, the natural number that we want to decompose into different Fibonacci numbers.

Base Case $n = 1$. Clearly true.

Inductive Hypothesis Assume that the claim is true for all natural numbers smaller than $n-1$.

Inductive Step Consider the largest Fibonacci number $\leq n$, which we will denote F_k. In another words, $F_k + F_{k-1} \geq n$. From there, we know then by simple algebra that $n - F_k \leq F_{k-1}$. Hence, $n - F_k$ is the sum of different Fibonacci numbers by the Inductive Hypothesis and furthermore, the sum cannot contain F_k so we see that n can also be written as a sum of different Fibonacci numbers.

6. False Proof

What goes wrong in the following "proof"?

Theorem: If n is an even number and $n \geq 2$, then n is a power of two.

Proof:
By induction on the natural number n. Let the induction hypothesis $IH(k)$ be the assertion that "if k is an even number and $k \geq 2$, then $k = 2^i$, where i is a natural number".

Base case: $IH(2)$ states that 2 is a power of two, which it is ($2 = 2^1$).

Inductive step: Assume that k is a number greater than 2, and that $IH(j)$ holds for all $2 \leq j < k$.

Case 1: k is odd, and there is nothing to show.

Case 2: k is even, so $k \geq 4$. Since $k \geq 4$ is an even number, $k = 2l$, with $2 \leq l < k$. Therefore we can use the induction hypothesis $IH(l)$, which asserts that $l = 2^i$ for some integer i. Thus we have $k = 2l = 2^{i+1}$, so k is a power of two. $IH(k)$ holds.

Answer:

The error in the proof is in the application of the induction hypothesis. The proof states that the induction hypotheses $IH(l)$ asserts that $l = 2i$, but in reality, it asserts that if l is even, then $l = 2i$. Since l may be odd, it is not possible to conclude that $l = 2i$.