1. The Central Limit Theorem.

Let us return to a previous online homework problem: recall the game in which I have a bag full of $1 and $5 bills, and each round you draw a single bill. I advertise that the average profit of a single round is $3, and that the variance is also 3 (Note: $\sqrt{3} \approx 1.732$).

a) Suppose you play 10 rounds and have an average profit $A_{10}$ such that $A_{10} \leq 1.40$. Using the central limit theorem, what is the approximate probability of this outcome for these 10 games?

b) Now, suppose you play 100 rounds and have an average profit $A_{100}$ such that $A_{100} \leq 1.40$. Using the central limit theorem, what is the order of magnitude of the probability of this outcome for these 100 games?

2. Chebyshev and Chernoff Bounds

Consider a biased coin with probability $p = 1/3$ of landing heads and probability $2/3$ of landing tails. Suppose the coin is flipped some number $n$ of times, and let $X_i$ be a random variable denoting the $i^{th}$ flip, where $X_i = 1$ means heads, and $X_i = 0$ means tails. Let random variable $X = \frac{1}{n} \sum_{i=1}^{n} X_i$. Compute the following expectation and variance:

a) What is $E[X_i]$?

b) What is $Var[X_i]$?

c) What is $E[X]$?

d) What is $Var[X]$?
Now we try to use both the Chebyshev's Inequality and the Chernoff Bound to determine a value for \( n \) so that the probability that more than half of the coin flips come out heads is less than 0.001.

e) The Chebyshev's Inequality says that for a random variable \( X \) with expectation \( \mathbb{E}[X] = \mu \), and for any \( \alpha > 0 \)

\[
\Pr[|X - \mu| \geq \alpha] \leq \frac{\text{Var}[X]}{\alpha^2} \tag{1}
\]

According to the definition of probability, we also have

\[
\Pr[X - \mu \geq \alpha] \leq \Pr[|X - \mu| \geq \alpha] \tag{2}
\]

To determine \( n \), what should \( \alpha \) be?

f) What is the minimum value of \( n \) according to the Chebyshev Inequality?

g) The Chernoff Inequality says if \( X_i \)s are i.i.d. and \( X = \frac{1}{n} \sum_{i=1}^{n} X_i \), then

\[
\Pr[X \geq \alpha] \leq e^{-n\Phi_{X_i}(\alpha)}, \text{ for } \alpha \geq p \text{ or } \Pr[X \leq \alpha] \leq e^{-n\Phi_{X_i}(\alpha)}, \text{ for } \alpha \leq p \tag{3}
\]

where \( \Phi_{X_i}(\alpha) \) is called the Kullback-Liebler Divergence, usually denoted by \( D(a||p) \)

\[
D(a||p) = \Phi_{X_i}(\alpha) = \alpha \ln \frac{\alpha}{p} + (1 - \alpha) \ln \frac{1 - \alpha}{1 - p} \tag{4}
\]

To determine \( n \), what should \( \alpha \) be?

h) What is the minimum value of \( n \) according to the Chernoff Bound?