Hypercubes

Recall that the set of all $n$-bit strings is denoted by $\{0,1\}^n$. The $n$-dimensional hypercube is a graph whose vertex set is $\{0,1\}^n$ (i.e. there are exactly $2^n$ vertices, each labeled with a distinct $n$-bit string), and with an edge between vertices $x$ and $y$ iff $x$ and $y$ differ in exactly one bit position. i.e. if $x = x_1x_2\ldots x_n$ and $y = y_1y_2\ldots y_n$, then there is an edge between $x$ and $y$ iff there is an $i$ such that $\forall j \neq i, x_j = y_j$ and $x_i \neq y_i$.

There is another equivalent recursive definition of the hypercube:

The $n$-dimensional hypercube consists of two copies of the $n-1$-dimensional hypercube (the 0-subcube and the 1-subcube), and with edges between corresponding vertices in the two subcubes. i.e. there is an edge between vertex $x$ in the 0-subcube (also denoted as vertex 0$x$) and vertex $x$ in the 1-subcube.

**Claim:** The total number of edges in an $n$-dimensional hypercube is $2^{n-1}$.

**Proof:** Each vertex has $n$ edges incident to it, since there are exactly $n$ bit positions that can be toggled to get an edge. Since each edge is counted twice, once from each endpoint, this yields a grand total of $n2^n/2$.

**Alternative Proof:** By the second definition, it follows that $E(n) = 2E(n-1) + 2^{n-1}$, and $E(1) = 1$. A straightforward induction shows that $E(n) = n2^{n-1}$.

We will prove that the $n$-dimensional hypercube is a very robust graph in the following sense: consider how many edges must be cut to separate a subset $S$ of vertices from the remaining vertices $V - S$. Assume that $S$ is the smaller piece; i.e. $|S| \leq |V - S|$.

**Theorem:** $|E_{S,V-S}| \geq |S|$.

**Proof:** By induction on $n$. Base case $n = 1$ is trivial.

For the induction step, let $S_0$ be the vertices from the 0-subcube in $S$, and $S_1$ be the vertices in $S$ from the 1-subcube.

Case 1: If $|S_0| \leq 2^{n-1}/2$ and $|S_1| \leq 2^{n-1}/2$ then applying the induction hypothesis to each of the subcubes shows that the number of edges between $S$ and $V - S$ even without taking into consideration edges that cross between the 0-subcube and the 1-subcube, already exceed $|S_0| + |S_1| = |S|$.

Case 2: Suppose $|S_0| > 2^{n-1}/2$. Then $|S_1| \leq 2^{n-1}/2$. But now $|E_{S,V-S}| \geq 2^n - 1 \geq |S|$. This is because by the induction hypothesis, the number of edges in $E_{S,V-S}$ within the 0-subcube is at least $2^{n-1} - |S_0|$, and those within the 1-subcube is at least $|S_1|$. But now there must be at least $|S_0| - |S_1|$ edges in $E_{S,V-S}$ that cross between the two subcubes (since there are edges between every pair of corresponding vertices. This is a grand total of $2^n - 1 - |S_0| + |S_1| + |S_0| - |S_1| = 2^{n-1}$. 