

Problem Set 3

1. **Simple Induction** Prove that for every $n \geq 2$, $\sqrt{1 + \sqrt{1 + \cdots \sqrt{1}}}$ (n square roots) is irrational.
2. **Strengthen induction hypothesis** Prove that for all $n \geq 1$, all the entries of the matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^n$ are bounded above by n .
3. **Proofs to Grade**
Grade the following induction proofs. If you think the proof is incorrect, please explain exactly what is wrong with the structure or the reasoning in the *proof*. Remember, simply saying that the claim (or the induction hypothesis is false) is *not* a justification.

(a) **Claim:** $(\forall n \in \mathcal{N})(n^2 \leq n)$. **Proof:**

- i) Base Case: When $n = 1$, the statement is $1^2 \leq 1$ which is true.
- ii) Induction hypothesis: Assume that $k^2 \leq k$.
- ii) Inductive step: We need to show that

$$(k + 1)^2 \leq k + 1$$

Working backwards we see that:

$$(k + 1)^2 \leq k + 1k^2 + 2k + 1 \leq k + 1k^2 + 2k \leq kk^2 \leq k$$

So we get back to our original hypothesis which is assumed to be true. Hence, for every $n \in \mathcal{N}$ we know that $n^2 \leq n$. ♠

(b) **Claim:** $(\forall n \in \mathcal{N})(7^n = 1)$.

Proof: (uses strong induction)

Base Case: Certainly $7^0 = 1$.

Induction hypothesis: Assume that $7^j = 1$ for all $0 \leq j \leq k$.

Inductive step: We need to prove that $7^{k+1} = 1$. But,

$$7^{k+1} = \frac{(7^k \cdot 7^k)}{7^{k-1}} = \frac{(1 \cdot 1)}{1} = 1$$

Hence, by the Principle of Strong Induction, $7^m = 1$ for all $m \in \mathcal{N}$. ♡

(c) **Claim:** For all natural numbers $n \geq 4$, $2^n < n!$.

Proof: $2^4 = 16$ and $4! = 24$, so the statement is true for $n = 4$.

Assume that $2^n < n!$ for some $n \in \mathcal{N}$. Then $2^{n+1} = 2(2^n) < 2(n!) \leq (n+1)(n!) = (n+1)!$, so $2^{n+1} < (n+1)!$. By the principle of mathematical induction, the statement is true for all $n \geq 4$.

♣

(d) **Claim:** All natural numbers are equal.

Proof: It is sufficient to show that for any two natural numbers A and B, $A = B$. Further, it is sufficient to show that for all $N \geq 0$, if A and B satisfy $\text{MAX}(A, B) = N$ then $A = B$.

Proceed by induction on N .

Base case: If $N = 0$, then A and B, being natural numbers, must both be 0. So $A = B$.

Induction Step: Assume that the theorem is true for some value k. Take A and B with $\text{MAX}(A, B) = k+1$. Then $\text{MAX}((A-1), (B-1)) = k$. And hence $(A-1) = (B-1)$. Consequently, $A = B$. \diamond

4. Consider the following recursively program for a function f :

```
function f(n)
  if n = 0 return 0
  else return(f(n - 1) + 2n)
end function
```

Prove by induction that $f(n) = n^2 + n$.

5. Examples for stable marriage algorithm.

(a) For the following instance of the stable marriage problem:

Men's preference list:

1	B	A	C
2	C	B	A
3	A	C	B

Women's preference list:

A	1	2	3
B	2	3	1
C	3	1	2

- i. List the rogue couples in the pairing $(1, C), (2, B), (3, A)$.
- ii. List all the stable pairings.

(b) Run the traditional propose and reject algorithm on the following example.

Men's preference list:

1	A	B	C	D
2	B	C	A	D
3	C	A	B	D
4	A	B	C	D

Women's preference list:

A	2	3	4	1
B	3	4	1	2
C	4	1	2	3
D	1	2	3	4