Hypercubes

Recall that the set of all n-bit strings is denoted by $\{0,1\}^n$. The n-dimensional hypercube is a graph whose vertex set is $\{0,1\}^n$ (i.e. there are exactly 2^n vertices, each labeled with a distinct n-bit string), and with an edge between vertices x and y iff x and y differ in exactly one bit position. i.e. if $x = x_1x_2\ldots x_n$ and $y = y_1y_2\ldots y_n$, then there is an edge between x and y iff there is an i such that $\forall j \neq i, x_j = y_j$ and $x_i \neq y_i$.

There is another equivalent recursive definition of the hypercube:

The n-dimensional hypercube consists of two copies of the $n-1$-dimensional hypercube (the 0-subcube and the 1-subcube), and with edges between corresponding vertices in the two subcubes. i.e. there is an edge between vertex x in the 0-subcube (also denoted as vertex $0x$) and vertex x in the 1-subcube.

Claim: The total number of edges in an n-dimensional hypercube is $n2^{n-1}$.

Proof: Each vertex has n edges incident to it, since there are exactly n bit positions that can be toggled to get an edge. Since each edge is counted twice, once from each endpoint, this yields a grand total of $n2^n/2$.

Alternative Proof: By the second definition, it follows that $E(n) = 2E(n-1) + 2^n$, and $E(1) = 1$. A straightforward induction shows that $E(n) = n2^{n-1}$.

We will prove that the n-dimensional hypercube is a very robust graph in the following sense: consider how many edges must be cut to separate a subset S of vertices from the remaining vertices $V-S$. Assume that S is the smaller piece; i.e. $|S| \leq |V-S|$.

Theorem: $|E_{S,V-S}| \geq |S|$.

Proof: By induction on n. Base case $n = 1$ is trivial.

For the induction step, let S_0 be the vertices from the 0-subcube in S, and S_1 be the vertices in S from the 1-subcube.

Case 1: If $|S_0| \leq 2^{n-1}/2$ and $|S_1| \leq 2^{n-1}/2$ then applying the induction hypothesis to each of the subcubes shows that the number of edges between S and $V-S$ even without taking into consideration edges that cross between the 0-subcube and the 1-subcube, already exceed $|S_0| + |S_1| = |S|$.

Case 2: Suppose $|S_0| > 2^{n-1}/2$. Then $|S_1| \leq 2^{n-1}/2$. But now $|E_{S,V-S}| \geq 2^n - 1 \geq |S|$. This is because by the induction hypothesis, the number of edges in $E_{S,V-S}$ within the 0-subcube is at least $2^{n-1} - |S_0|$, and those within the 1-subcube is at least $|S_1|$. But now there must be at least $|S_0| - |S_1$ edges in $E_{S,V-S}$ that cross between the two subcubes (since there are edges betwen every pair of corresponding vertices. This is a grand total of $2^{n-1} - |S_0| + |S_1| + |S_0| - |S_1| = 2^{n-1}$.