A Recursive Structure

- Trees naturally represent recursively defined, hierarchical objects with more than one recursive subpart for each instance.

- Common examples: expressions, sentences.
  - Expressions have definitions such as “an expression consists of a literal or two expressions separated by an operator.”

- Also describe structures in which we recursively divide a set into multiple subsets.
Formal Definitions

- Trees come in a variety of flavors, all defined recursively:
  - **61A style**: A tree consists of a *label* value and zero or more *branches* (or *children*), each of them a tree.
  - **61A style, alternative definition**: A tree is a set of *nodes* (or *vertices*), each of which has a label value and one or more *child nodes*, such that no node descends (directly or indirectly) from itself. A node is the *parent* of its children.
  - **Positional trees**: A tree is either *empty* or consists of a node containing a label value and an indexed sequence of zero or more children, each a positional tree. If every node has two positions, we have a *binary tree* and the children are its *left and right sub-trees*. Again, nodes are the parents of their non-empty children.
  - We’ll see other varieties when considering graphs.
Tree Characteristics (I)

- The **root** of a tree is a non-empty node with no parent in that tree (its parent might be in some larger tree that contains that tree as a subtree). Thus, every node is the root of a (sub)tree.

- The **order**, **arity**, or **degree** of a node (tree) is its number (maximum number) of children.

- The nodes of a **k-ary tree** each have at most $k$ children.

- A **leaf** node has no children (no non-empty children in the case of positional trees).
Tree Characteristics (II)

- The **height** of a node in a tree is the smallest distance to a leaf. That is, a leaf has height 0 and a non-empty tree's height is one more than the maximum height of its children. The height of a tree is the height of its root.

- The **depth** of a node in a tree is the distance to the root of that tree. That is, in a tree whose root is $R$, $R$ itself has depth 0 in $R$, and if node $S \neq R$ is in the tree with root $R$, then its depth is one greater than its parent's.
A Tree Type, 61A Style

public class Tree<Label> {

    // This constructor is convenient, but unfortunately requires this
    // SuppressWarnings annotation to prevent (harmless) warnings
    // that we will explain later.
    @SuppressWarnings("unchecked")
    public Tree(Label label, Tree<Label>... children) {
        _label = label;
        _kids = new ArrayList<>((Arrays.asList(children));
    }

    public int arity() { return _kids.size(); }

    public Label label() { return _label; }

    public Tree<Label> child(int k) { return _kids.get(k); }

    private Label _label;
    private ArrayList<Tree<Label>> _kids;
}
Fundamental Operation: Traversal

- **Traversing a tree** means enumerating (some subset of) its nodes.
- Typically done recursively, because that is natural description.
- As nodes are enumerated, we say they are **visited**.
- Three basic orders for enumeration (+ variations):
  - **Preorder**: visit node, traverse its children.
  - **Postorder**: traverse children, visit node.
  - **Inorder**: traverse first child, visit node, traverse second child (binary trees only).

![Tree Traversal Examples](image)
Preorder Traversal and Prefix Expressions

Problem: Convert 

- * x + y

3 z

into

(- (- (* x (+ y 3))) z)

(Assume Tree<Label> is means "Tree whose labels have type Label."

static String toLisp(Tree<String> T) {
    if (T.arity() == 0) return T.label();
    else {
        String R; R = "(" + T.label();
        for (int i = 0; i < T.arity(); i += 1)
            R += " " + toLisp(T.child(i));
        return R + ")";
    }
}
Inorder Traversal and Infix Expressions

Problem: Convert

- 
  -
  *
  
  x
  +
  y
  3

into

\((-x \cdot (y + 3)) - z\)

To think about: how to get rid of all those parentheses.

static String toInfix(Tree<String> T) {
    if (T.arity() == 0) {
        return T.label();
    } else if (T.arity() == 1) {
        return "(" + T.label() + toInfix(T.child(0)) + ")";
    } else {
        return "(" + toInfix(T.child(0)) + T.label() + toInfix(T.child(1)) + ")";
    }
}
Postorder Traversal and Postfix Expressions

Problem: Convert

\[- \frac{x \times y + 3}{z} \]

into

\[x \ y \ 3 \ +:2 \ *:2 \ -:1 \ z \ -:2\]

static String toPolish(Tree<String> T)
{
    String R; R = "";
    for (int i = 0; i < T.arity(); i += 1)
        R += toPolish(T.child(i)) + " ";
    return R + String.format("%s:%d", T.label(), T.arity());
}
A General Traversal: The Visitor Pattern

```java
void preorderTraverse(Tree<Label> T, Consumer<Tree<Label>> visit)
{
    if (T != null) {
        visit.accept(T);
        for (int i = 0; i < T.arity(); i += 1)
            preorderTraverse(T.child(i), visit);
    }
}
```

- `java.util.function.Consumer<AType>` is a library interface that works as a function-like type with one void method, `accept`, which takes an argument of type `AType`.

- Now, using Java 8 lambda syntax, I can print all labels in the tree in preorder with:

  ```java
  preorderTraverse(myTree, T -> System.out.print(T.label() + " "));
  ```
Iterative Depth-First Traversals

- Tree recursion conceals data: a stack of nodes (all the T arguments) and a little extra information. Can make the data explicit:

```java
void preorderTraverse2(Tree<Label> T, Consumer<Tree<Label>> visit) {
    Stack<Tree<Label>> work = new Stack<>();
    work.push(T);
    while (!work.isEmpty()) {
        Tree<Label> node = work.pop();
        visit.accept(node);
        for (int i = node.arity()-1; i >= 0; i -= 1)
            work.push(node.child(i)); // Why backward?
    }
}
```

- This traversal takes the same $\Theta(\cdot)$ time as doing it recursively, and also the same $\Theta(\cdot)$ space.

- That is, we have substituted an explicit stack data structure (work) for Java’s built-in execution stack (which handles function calls).
Level-Order (Breadth-First) Traversal

**Problem:** Traverse all nodes at depth 0, then depth 1, etc:

```
    0
   / \
  1   2
 / \ / \  \
3  4 5  6
```

Last modified: Sun Oct 13 16:46:16 2019
Breadth-First Traversal Implemented

A simple modification to iterative depth-first traversal gives breadth-first traversal. Just change the (LIFO) stack to a (FIFO) queue:

```java
void breadthFirstTraverse(Tree<Label> T, Consumer<Tree<Label>> visit) {
    ArrayDeque<Tree<Label>> work = new ArrayDeque<>();  // (Changed)
    work.push(T);
    while (!work.isEmpty()) {
        Tree<Label> node = work.remove();  // (Changed)
        if (node != null) {
            visit.accept(node);
            for (int i = 0; i < node.arity(); i += 1) // (Changed)
                work.push(node.child(i));
        }
    }
}
```
The traversal algorithms have roughly the form of the boom example in §1.3.3 of *Data Structures*—an exponential algorithm.

However, the role of $M$ in that algorithm is played by the *height* of the tree, not the number of nodes.

In fact, easy to see that tree traversal is *linear*: $\Theta(N)$, where $N$ is the # of nodes: Form of the algorithm implies that there is one visit at the root, and then one visit for every *edge* in the tree. Since every node but the root has exactly one parent, and the root has none, must be $N - 1$ edges in any non-empty tree.

In positional tree, is also one recursive call for each empty tree, but # of empty trees can be no greater than $kN$, where $k$ is arity.

For $k$-ary tree (max # children is $k$), $h + 1 \leq N \leq \frac{k^{h+1} - 1}{k - 1}$, where $h$ is height.

So $h \in \Omega(\log_k N) = \Omega(\lg N)$ and $h \in O(N)$.

Many tree algorithms look at one child only. For them, worst-case time is proportional to the *height* of the tree—$\Theta(\lg N)$—assuming that tree is *bushy*—each level has about as many nodes as possible.
Recursive Breadth-First Traversal: Iterative Deepening

- Previous breadth-first traversal used space proportional to the width of the tree, which is $\Theta(N)$ for bushy trees, whereas depth-first traversal takes $\lg N$ space on bushy trees.
- Can we get breadth-first traversal in $\lg N$ space and $\Theta(N)$ time on bushy trees?
- For each level, $k$, of the tree from 0 to $\text{lev}$, call doLevel(T, k):

  ```java
  void doLevel(Tree T, int lev) {
    if (lev == 0)
      visit T
    else
      for each non-null child, C, of T {
        doLevel(C, lev-1);
      }
  }
  ```

- So we do breadth-first traversal by repeated (truncated) depth-first traversals: *iterative deepening*.
- In doLevel(T, k), we skip (i.e., traverse but don’t visit) the nodes before level $k$, and then visit at level $k$, but not their children.
Iterative Deepening Time?

- Let $h$ be height, $N$ be # of nodes.
- Count # edges traversed (i.e, # of calls, not counting null nodes).
- First (full) tree: 1 for level 0, 3 for level 1, 7 for level 2, 15 for level 3.
- Or in general $2^1 - 1 + 2^2 - 1 + \ldots + (2^{h+1} - 1) = 2^{h+2} - h \in \Theta(N)$, since $N = 2^{h+1} - 1$ for this tree.
- Second (right leaning) tree: 1 for level 0, 2 for level 2, 3 for level 3.
- Or in general $(h + 1)(h + 2)/2 = N(N + 1)/2 \in \Theta(N^2)$, since $N = h + 1$ for this kind of tree.
Iterators for Trees

- Frankly, iterators are not terribly convenient on trees.
- But can use ideas from iterative methods.

```java
class PreorderTreeIterator<Label> implements Iterator<Label> {
    private Stack<Tree<Label>> s = new Stack<Tree<Label>>();

    public PreorderTreeIterator(Tree<Label> T) { s.push(T); }

    public boolean hasNext() { return !s.isEmpty(); }
    public T next() {
        Tree<Label> result = s.pop();
        for (int i = result.arity()-1; i >= 0; i -= 1)
            s.push(result.child(i));
        return result.label();
    }
}
```

**Example: (what do I have to add to class Tree first?)**

```java
for (String label : aTree) System.out.print(label + " ");
```
Tree Representation

(a) Embedded child pointers (+ optional parent pointers)

(b) Array of child pointers (+ optional parent pointers)

(c) child/sibling pointers

(d) breadth-first array (complete trees)