CS61B Lecture #25

Today: Sorting, cont.

- Standard methods
- Properties of standard methods
- Selection

Readings for Today: *DS(IJ)*, Chapter 8;

Readings for Next Topic: Balanced searches, *DS(IJ)*, Chapter 9;

Shell's sort

Idea: Improve insertion sort by first sorting *distant* elements:

- First sort subsequences of elements $2^k - 1$ apart:
  - sort items #$0,\ 2^k - 1,\ 2(2^k - 1),\ 3(2^k - 1),\ \ldots$, then
  - sort items #$1,\ 1 + 2^k - 1,\ 1 + 2(2^k - 1),\ 1 + 3(2^k - 1),\ \ldots$, then
  - sort items #$2,\ 2 + 2^k - 1,\ 2 + 2(2^k - 1),\ 2 + 3(2^k - 1),\ \ldots$, then
  - *etc.*
  - sort items #$2^k - 2,\ 2(2^k - 1) - 1,\ 3(2^k - 1) - 1,\ \ldots$,
  - Each time an item moves, can reduce #inversions by as much as $2^k + 1$.
- Now sort subsequences of elements $2^{k-1} - 1$ apart:
  - sort items #$0,\ 2^{k-1} - 1,\ 2(2^{k-1} - 1),\ 3(2^{k-1} - 1),\ \ldots$, then
  - sort items #$1,\ 1 + 2^{k-1} - 1,\ 1 + 2(2^{k-1} - 1),\ 1 + 3(2^{k-1} - 1),\ \ldots$,
  - $\vdots$
- End at plain insertion sort ($2^0 = 1$ apart), but with most inversions gone.
- Sort is $\Theta(N^{1.5})$ (take CS170 for why!).

Example of Shell's Sort

																#I	*#C*
15	14	13	12	11	10	9	8	7	6	5	4	3	2	1	0	120	1
0	14	13	12	11	10	9	8	7	6	5	4	3	2	1	15	91	10
0	7	6	5	4	3	2	1	14	13	12	11	10	9	8	15	42	20
0	1	3	2	4	6	5	7	8	10	9	11	13	12	14	15	4	19
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	0	-

I: Inversions left.

C: Comparisons needed to sort subsequences.

Sorting by Selection: Heapsort

Idea: Keep selecting smallest (or largest) element.

- Really bad idea on a simple list or vector.
- But we've already seen it in action: use heap.
- Gives $O(N \lg N)$ algorithm (N remove-first operations).
- Since we remove items from end of heap, we can use that area to accumulate result:

original:	19	0	-1	7	23	2	42	
heapified:	42	23	19	7	0	2	-1	
	23	7	19	-1	0	2		42
	19	7	2	-1	0		23	42
	7	0	2	-1		19	23	42
	2	0	-1		7	19	23	42
	0	-1		2	7	19	23	42
	-1		0	2	7	19	23	42

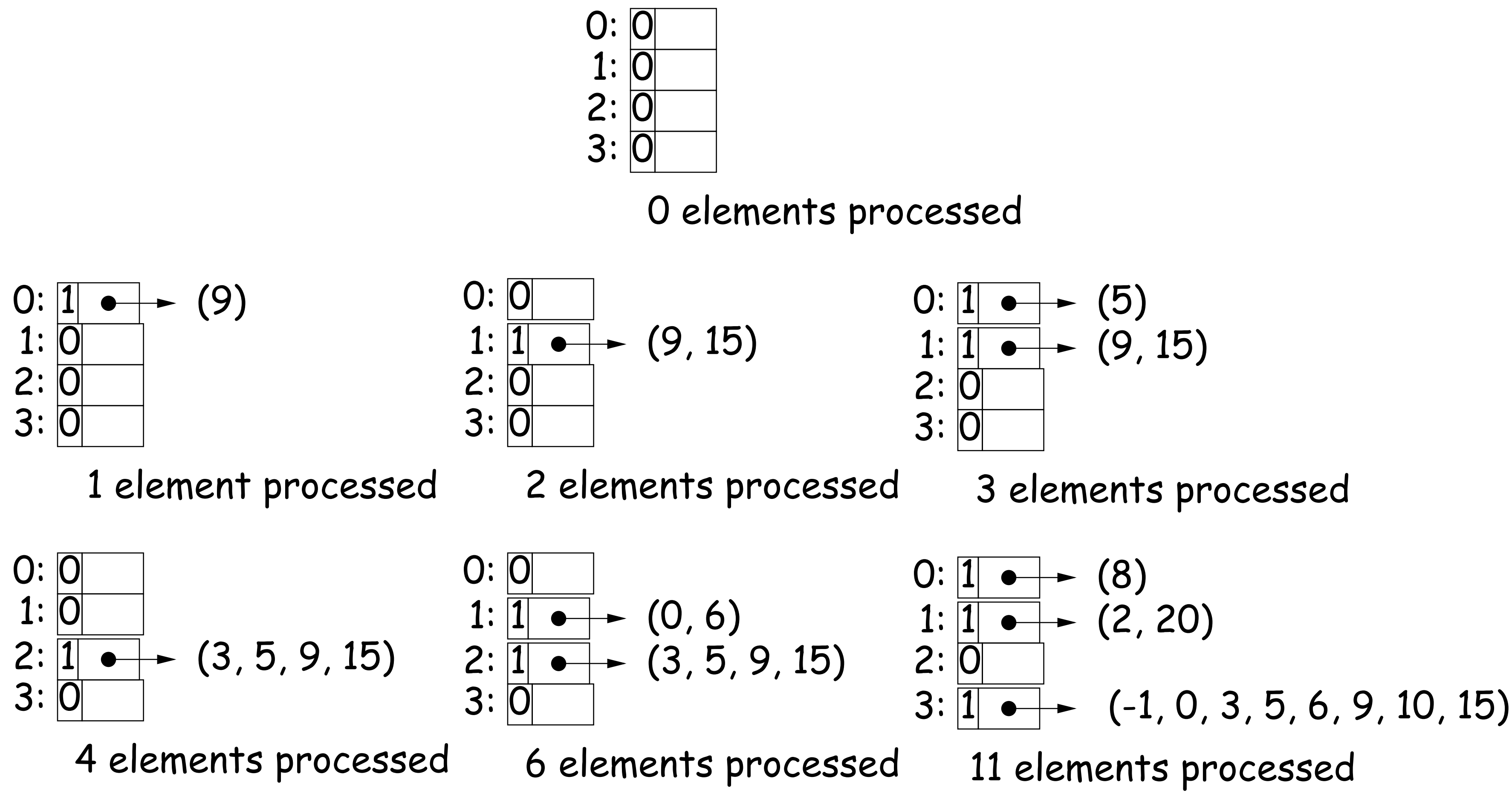
Merge Sorting

Idea: Divide data in 2 equal parts; recursively sort halves; merge results.

- Already seen analysis: $\Theta(N \lg N)$.
- Good for *external sorting:*
  - First break data into small enough chunks to fit in memory and sort.
  - Then repeatedly merge into bigger and bigger sequences.
  - Can merge K sequences of *arbitrary size* on secondary storage using $\Theta(K)$ storage.
- For internal sorting, can use *binomial comb* to orchestrate:

Illustration of Internal Merge Sort

L: (9, 15, 5, 3, 0, 6, 10, -1, 2, 20, 8)



Quicksort: Speed through Probability

Idea:

- *Partition* data into pieces: everything $>$ a *pivot* value at the high end of the sequence to be sorted, and everything $\leq$ on the low end.
- Repeat recursively on the high and low pieces.
- For speed, stop when pieces are "small enough" and do insertion sort on the whole thing.
- Reason: insertion sort has low constant factors. By design, no item will move out of its will move out of its piece [why?], so when pieces are small, #inversions is, too.
- Have to choose pivot well. E.g.: *median* of first, last and middle items of sequence.

Example of Quicksort

- In this example, we continue until pieces are size ≤ 4.
- Pivots for next step are starred. Arrange to move pivot to dividing line each time.
- Last step is insertion sort.

16 10 13 18 -4 -7 12 -5 19 15 0 22 29 34 -1*

-4 -5 -7 ‖ -1 ‖ 18 13 12 10 19 15 0 22 29 34 16*

-4 -5 -7 ‖ -1 ‖ 15 13 12* 10 0 ‖ 16 ‖ 19* 22 29 34 18

-4 -5 -7 ‖ -1 ‖ 10 0 ‖ 12 ‖ 15 13 ‖ 16 ‖ 18 ‖ 19 ‖ 29 34 22

- Now everything is "close to" right, so just do insertion sort:

-7 -5 -4 -1 0 10 12 13 15 16 18 19 22 29 34

Performance of Quicksort

- Probabalistic time:
  - If choice of pivots good, divide data in two each time: $\Theta(N \lg N)$ with a good constant factor relative to merge or heap sort.
  - If choice of pivots bad, most items on one side each time: $\Theta(N^2)$.
  - $\Omega(N \lg N)$ in best case, so insertion sort better for nearly ordered input sets.
- Interesting point: randomly shuffling the data before sorting makes $\Omega(N^2)$ time *very* unlikely!

Quick Selection

The Selection Problem: for given k, find k^{th} smallest element in data.

- Obvious method: sort, select element #k, time $\Theta(N \lg N)$.
- If $k \leq$ some constant, can easily do in $\Theta(N)$ time:
  - Go through array, keep smallest k items.
- Get *probably* $\Theta(N)$ *time* for all k by adapting quicksort:
  - Partition around some pivot, p, as in quicksort, arrange that pivot ends up at dividing line.
  - Suppose that in the result, pivot is at index m, all elements $\leq$ pivot have indicies $\leq m$.
  - If $m = k$, you're done: p is answer.
  - If $m > k$, recursively select k^{th} from left half of sequence.
  - If $m < k$, recursively select $(m - k - 1)^{\text{th}}$ from right half of sequence.

Selection Example

Problem: Find just item #10 in the sorted version of array:

Initial contents:

51	60	21	-4	37	4	49	10	40*	59	0	13	2	39	11	46	31
0																

Looking for #10 to left of pivot 40:

13	31	21	-4	37	4*	11	10	39	2	0	40	59	51	49	46	60
0																

Looking for #6 to right of pivot 4:

-4	0	2	4	37	13	11	10	39	21	31*	40	59	51	49	46	60
				4												

Looking for #1 to right of pivot 31:

-4	0	2	4	21	13	11	10	31	39	37	40	59	51	49	46	60
									9							

Just two elements; just sort and return #1:

-4	0	2	4	21	13	11	10	31	37	39	40	59	51	49	46	60
									9							

Result: 39

Selection Performance

- For this algorithm, if m roughly in middle each time, cost is

$$\begin{aligned} C(N) &= \begin{cases} 1, & \text{if } N = 1, \\ N + C(N/2), & \text{otherwise.} \end{cases} \\ &= N + N/2 + \ldots + 1 \\ &= 2N - 1 \in \Theta(N) \end{aligned}$$

- But in worst case, get $\Theta(N^2)$, as for quicksort.
- By another, non-obvious algorithm, can get $\Theta(N)$ worst-case time for all k (take CS170).

Better than N lg N?

- Can prove that *if all you can do to keys is compare them* then sorting must take $\Omega(N \lg N)$.
- Basic idea: there are $N!$ possible ways the input data could be scrambled.
- Therefore, your program must be prepared to do $N!$ different combinations of move operations.
- Therefore, there must be $N!$ possible combinations of outcomes of all the **if** tests in your program (we're assuming that comparisons are 2-way).
- Since each **if** test goes two ways, number of possible different outcomes for k **if** tests is 2^k.
- Thus, need enough tests so that $2^k > N!$, which means $k \in \Omega(\lg N!)$.
- Using Stirling's approximation,

$$m! \in \sqrt{2\pi m}\left(\frac{m}{e}\right)^m \left(1 + \Theta\left(\frac{1}{m}\right)\right),$$

this tells us that

$$k \in \Omega(N \lg N).$$

Beyond Comparison: Distribution Counting

- But suppose can do more than compare keys?
- For example, how can we sort a set of N integer keys whose values range from 0 to kN, for some small constant k?
- One technique: *count* the number of items < 1, < 2, etc.
- If $M_p =$#items with value $< p$, then in sorted order, the j^{th} item with value p must be #$M_p + j$.
- Gives *linear-time* algorithm.

Distribution Counting Example

- Suppose all items are between 0 and 9 as in this example:

7	0	4	0	9	1	9	1	9	5	3	7	3	1	6	7	4	2	0

3	3	1	2	2	1	1	3	0	3	*Counts*
0	1	2	3	4	5	6	7	8	9	

0	3	6	7	9	11	12	13	16	16	*Running sum*
<0	<1	<2	<3	<4	<5	<6	<7	<8	<9	

0	0	0	1	1	1	2	3	3	4	4	5	6	7	7	7	9	9	9
0			3			6			9		11	12	13			16		

- "Counts" line gives # occurrences of each key.
- "Running sum" gives cumulative count of keys $\leq$ each value...
- ...which tells us where to put each key:
- The first instance of key k goes into slot m, where m is the number of key instances that are $< k$.

Radix Sort

Idea: *Sort keys one character at a time.*

- Can use distribution counting for each digit.
- Can work either right to left (LSD radix sort) or left to right (MSD radix sort)
- LSD radix sort is venerable: used for punched cards.

Initial: set, cat, cad, con, bat, can, be, let, bet

Pass 1 (by char #2)

'␣'	'd'	'n'	't'
			bet
			let
			bat
		can	cat
be	cad	con	set

be, cad, con, can, set, cat, bat, let, bet

Pass 2 (by char #1)

'a'	'e'	'o'
bat	bet	
cat	let	
can	set	
cad	be	con

cad, can, cat, bat, be, set, let, bet, con

Pass 3 (by char #0)

'b'	'c'	'l'	's'
	con		
bet	cat		
be	can		
bat	cad	let	set

bat, be, bet, cad, can, cat, con, let, set

MSD Radix Sort

- A bit more complicated: must keep lists from each step separate
- But, can stop processing 1-element lists

A	posn
$\star$ set, cat, cad, con, bat, can, be, let, bet	0
$\star$ bat, be, bet / cat, cad, con, can / let / set	1
bat / $\star$ be, bet / cat, cad, con, can / let / set	2
bat / be / bet / $\star$ cat, cad, con, can / let / set	1
bat / be / bet / $\star$ cat, cad, can / con / let / set	2
bat / be / bet / cad / can / cat / con / let / set	

Performance of Radix Sort

- Radix sort takes $\Theta(B)$ time where B is *total size of the key data*.
- Have measured other sorts as function of #records.
- How to compare?
- To have N different records, must have keys at least $\Theta(\lg N)$ long [why?]
- Furthermore, comparison actually takes time $\Theta(K)$ where K is size of key in worst case [why?]
- So $N \lg N$ comparisons really means $N(\lg N)^2$ operations.
- While radix sort takes $B = N \lg N$ time.
- On the other hand, must work to *get* good constant factors with radix sort.

And Don't Forget Search Trees

Idea: A search tree is in sorted order, when read in inorder.

- Need *balance* to really use for sorting [next topic].
- Given balance, same performance as heapsort: N insertions in time $\lg N$ each, plus $\Theta(N)$ to traverse, gives

$$\Theta(N + N \lg N) = \Theta(N \lg N)$$

Summary

- Insertion sort: $\Theta(Nk)$ comparisons and moves, where k is maximum amount data is displaced from final position.
  - Good for small datasets or almost ordered data sets.
- Quicksort: $\Theta(N \lg N)$ with good constant factor if data is not pathological. Worst case $O(N^2)$.
- Merge sort: $\Theta(N \lg N)$ guaranteed. Good for external sorting.
- Heapsort, treesort with guaranteed balance: $\Theta(N \lg N)$ guaranteed.
- Radix sort, distribution sort: $\Theta(B)$ (number of bytes). Also good for external sorting.