Higher-Order Functions

Announcements

Example: Prime Factorization

Prime Factorization

Each positive integer n has a set of prime factors: primes whose product is n

...
8 = 2 * 2 * 2
9 = 3 * 3
10 = 2 * 5
11 = 11
12 = 2 * 2 * 3
...

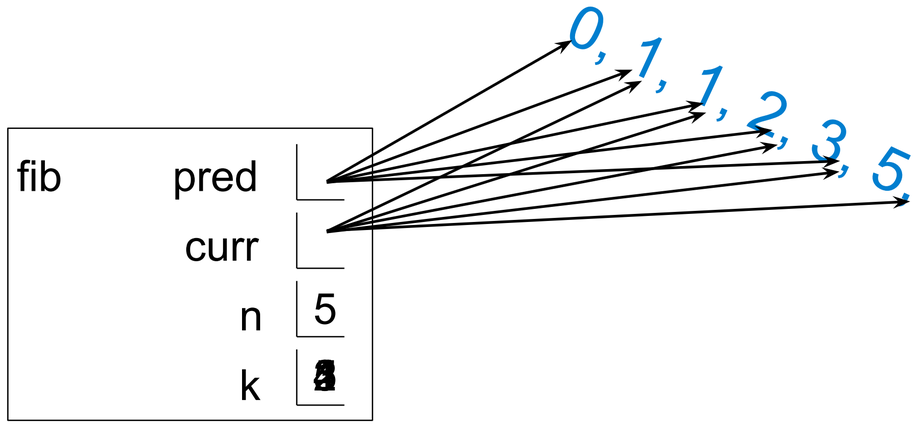
One approach: Find the smallest prime factor of n, then divide by it

858 = 2 * 429 = 2 * 3 * 143 = 2 * 3 * 11 * 13

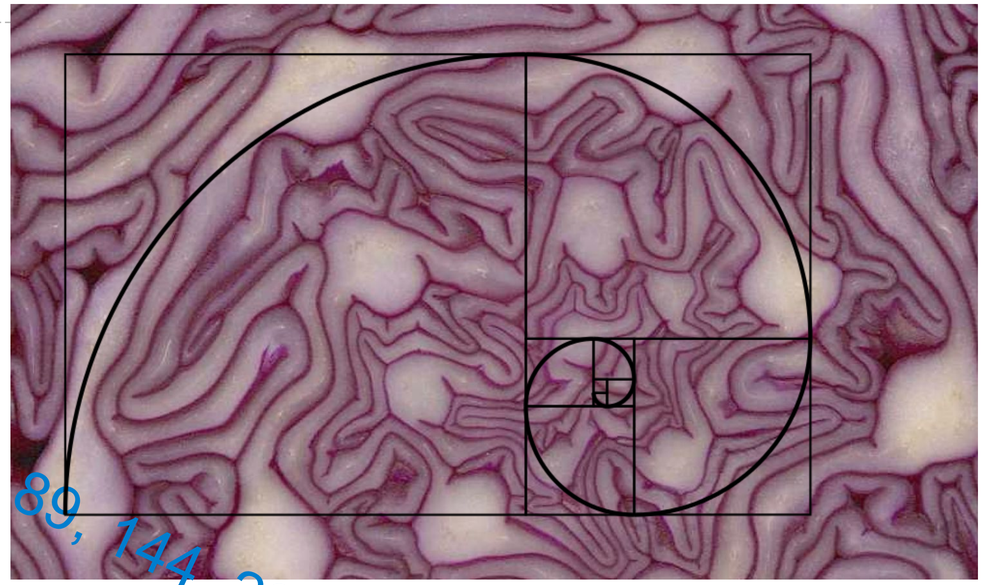
(Demo)

Example: Iteration

The Fibonacci Sequence



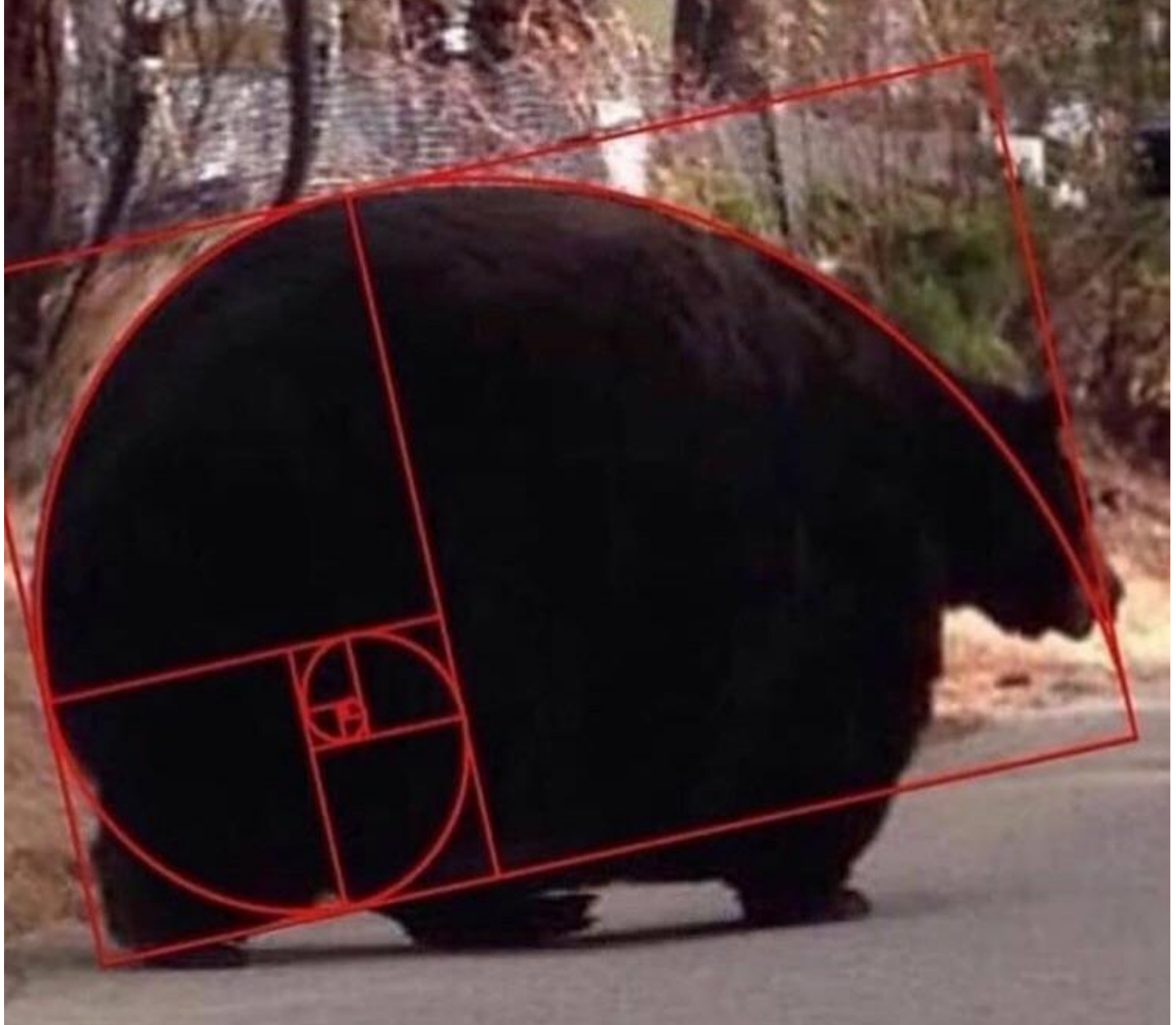
0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987



```
def fib(n):
    """Compute the nth Fibonacci number, for N >= 1."""
    pred, curr = 0, 1   # 0th and 1st Fibonacci numbers
    k = 1               # curr is the kth Fibonacci number
    while k < n:
        pred, curr = curr, pred + curr
        k = k + 1
    return curr
```

The next Fibonacci number is the sum of the current one and its predecessor

Go Bears!



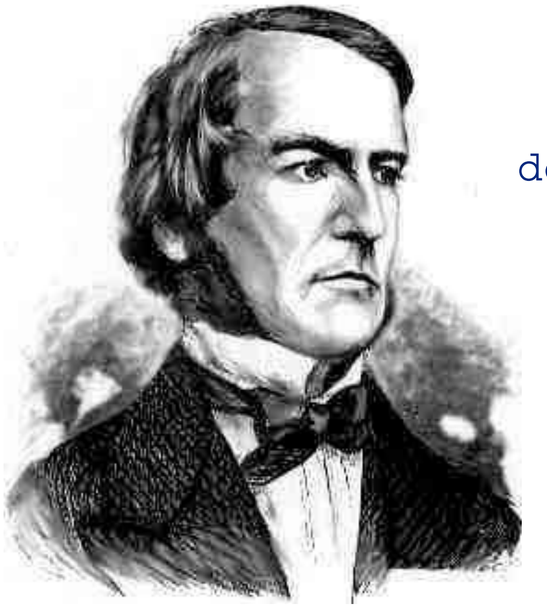
Control and Call Expressions

Boolean Contexts

George Boole

```
def absolute_value(x):
    """Return the absolute value of x."""
    if x < 0:
        return -x
    elif x == 0:
        return 0
    else:
        return x
```

Boolean Contexts



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```
def absolute_value(x):
    """Return the absolute value of x."""
    if x < 0:
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        return 0
    else:
        return x
```

Two boolean contexts

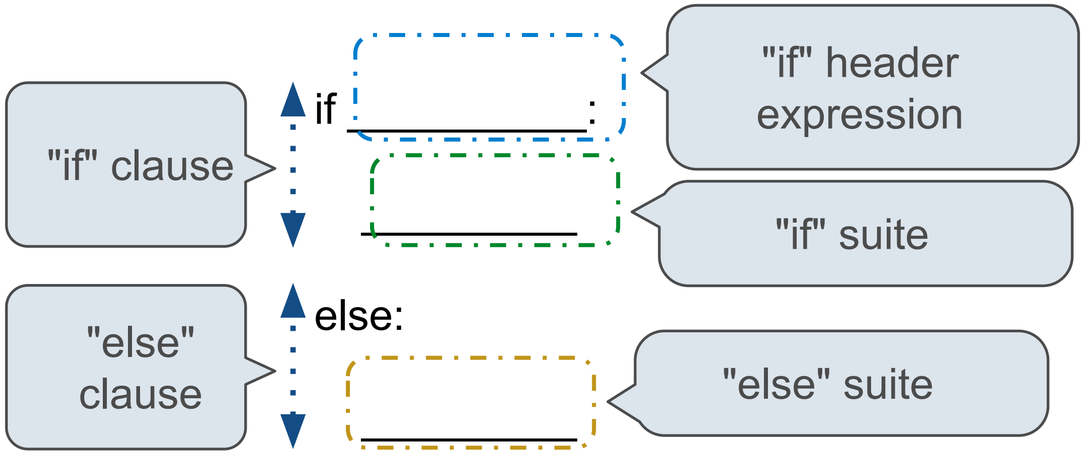
False values in Python: False, 0, '', None *(more to come)*

True values in Python: Anything else (True)

(Demo)

If Statements and Call Expressions

Let's try to write a function that does the same thing as an if statement.



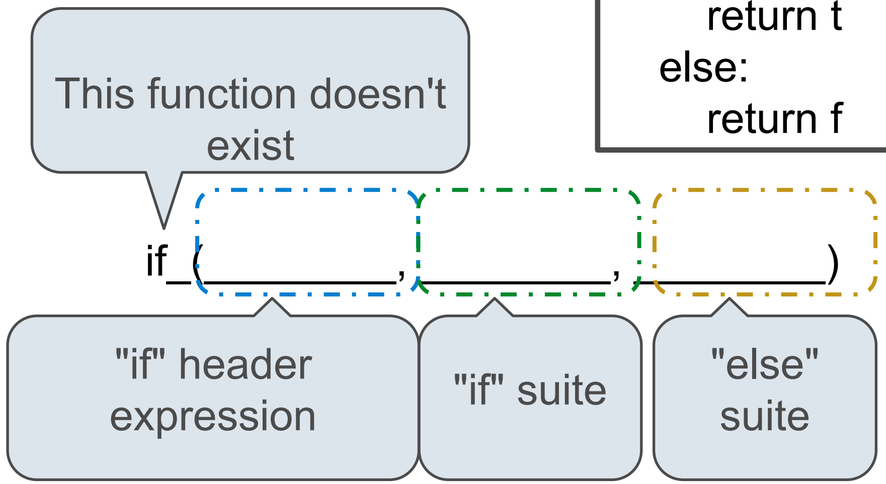
Execution Rule for Conditional Statements:

Each clause is considered in order.

1. Evaluate the header's expression (if present).
2. If it is a true value (or an else header), execute the suite & skip the remaining clauses.

(Demo)

```
def if_(c, t, f):
    if c:
        return t
    else:
        return f
```



Evaluation Rule for Call Expressions:

1. Evaluate the operator and then the operand subexpressions
2. Apply the function that is the value of the operator to the arguments that are the values of the operands

Higher-Order Functions

Generalizing Over Computational Processes

The common structure among functions may be a computational process, rather than a number.

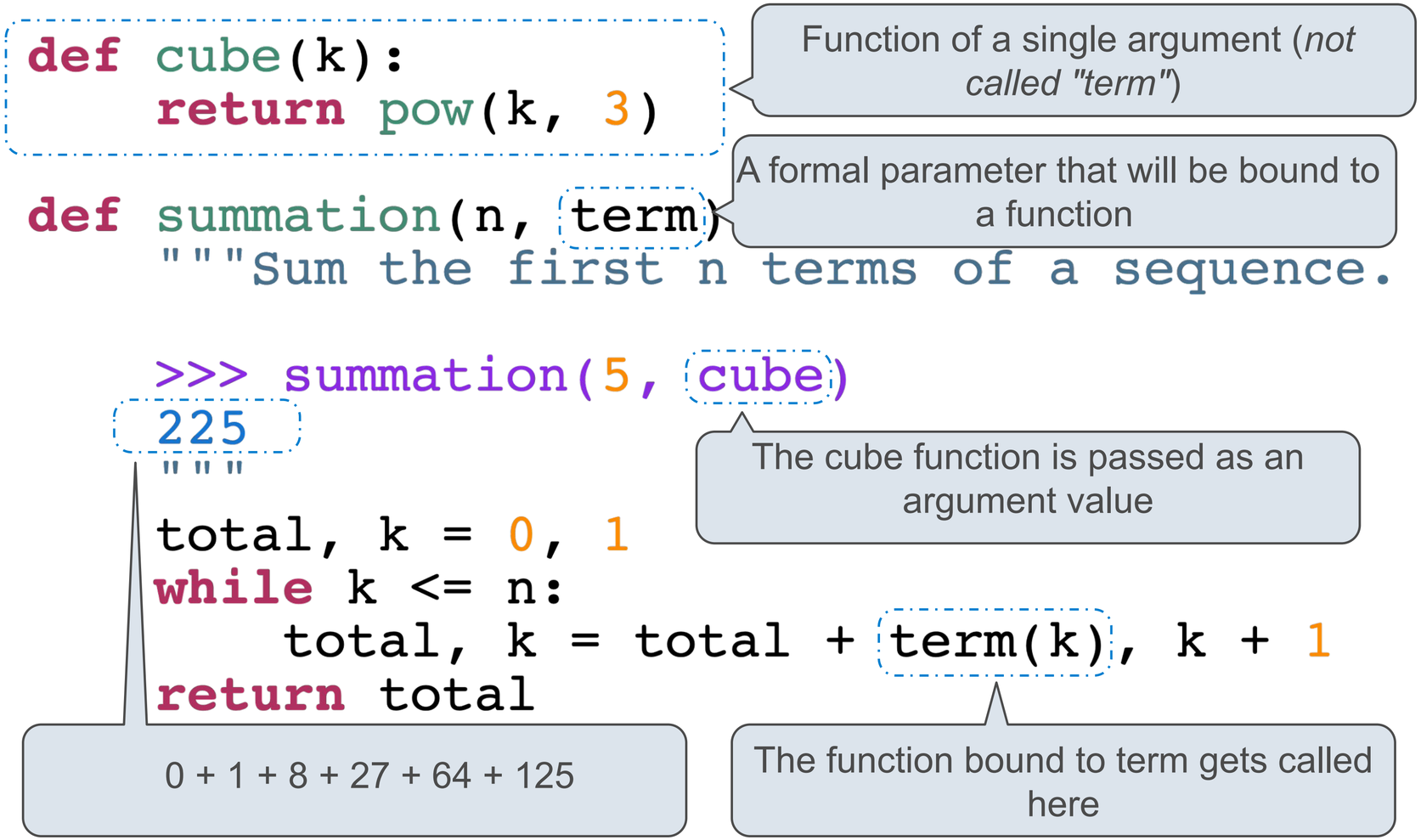
$$\sum_{k=1}^{5} k = 1 + 2 + 3 + 4 + 5 = 15$$

$$\sum_{k=1}^{5} k^3 = 1^3 + 2^3 + 3^3 + 4^3 + 5^3 = 225$$

$$\sum_{k=1}^{5} \frac{8}{(4k-3)\cdot(4k-1)} = \frac{8}{3} + \frac{8}{35} + \frac{8}{99} + \frac{8}{195} + \frac{8}{323} = 3.04$$

(Demo)

Summation Example



Break

Types of Higher-Order Functions

Environments Enable Higher-Order Functions

Functions are first-class: Functions are values in our programming language

Higher-order function: A function that takes a function as an argument value **or**
A function that returns a function as a return value

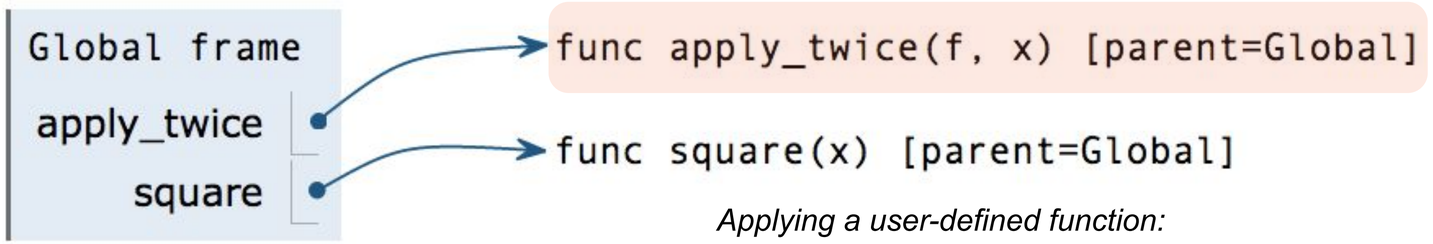
(Demo)

Environments for Higher-Order Functions

(Demo)

Names can be Bound to Functional Arguments

```
1  def apply_twice(f, x):
2      return f(f(x))
3
4  def square(x):
5      return x * x
6
7  result = apply_twice(square, 2)
```



Applying a user-defined function:

- Create a new frame
- Bind formal parameters (f & x) to arguments
- Execute the body: return f(f(x))

```
1  def apply_twice(f, x):
2      return f(f(x))
3
4  def square(x):
5      return x * x
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7  result = apply_twice(square, 2)
```
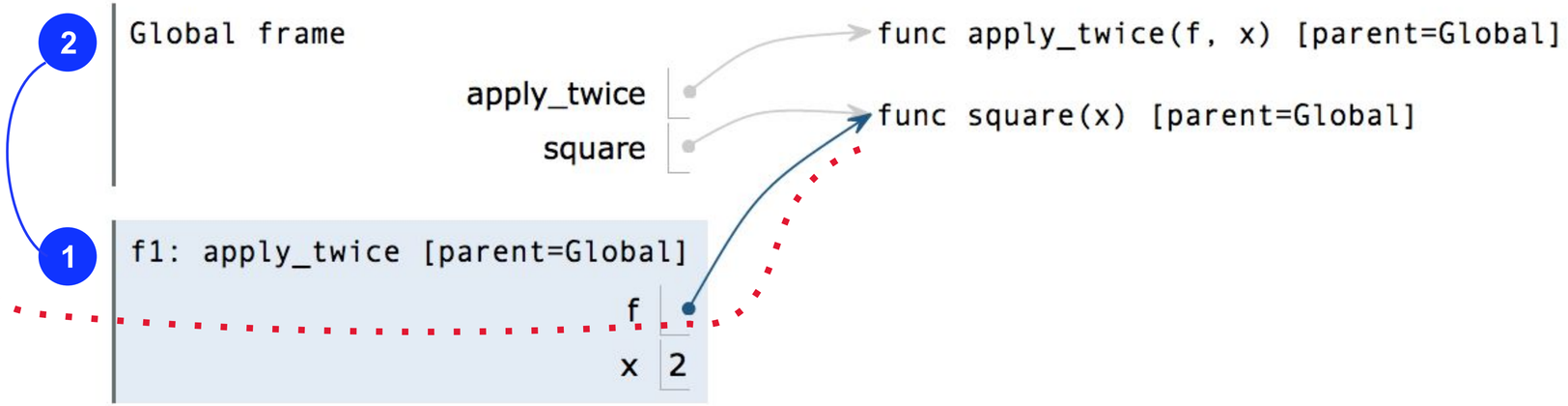



pythontutor.com/composingprograms.html#code=def%20apply_twice%28f,%20x%29%3A%0A%20%20%20%20return%20f%28f%28x%29%29%0A%20%20%20%20%0Adef%20square%28x%29%3A%0A%20%20%20%20return%20x%20*%20x%0A%20%20%20%20%0Aresult%20%3D%20apply_twice%28square,%202%29&mode=display&origin=composingprograms.js&cumulative=true&py=3&rawInputLstJSON=[]&curInstr=0

Functions as Return Values

(Demo)

Locally Defined Functions

Functions defined within other function bodies are bound to names in a local frame

A function that returns a function

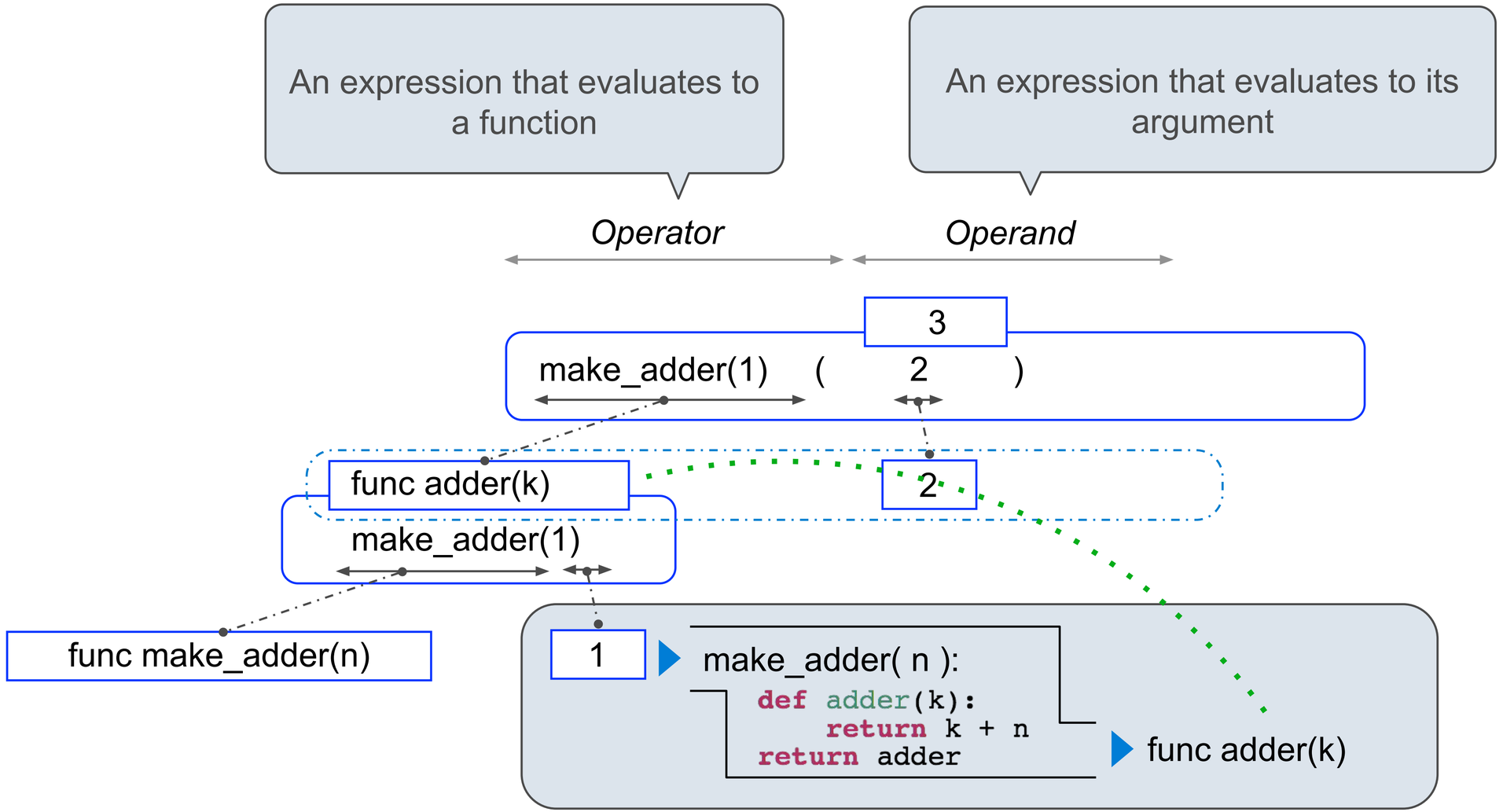
```
def make_adder(n):
    """Return a function that takes one argument k and returns k + n.

    >>> add_three = make_adder(3)
    >>> add_three(4)
    7
    """
    def adder(k):
        return k + n
    return adder
```

The name add_three is bound to a function

A def statement within another def statement

Can refer to names in the enclosing function

Call Expressions as Operator Expressions



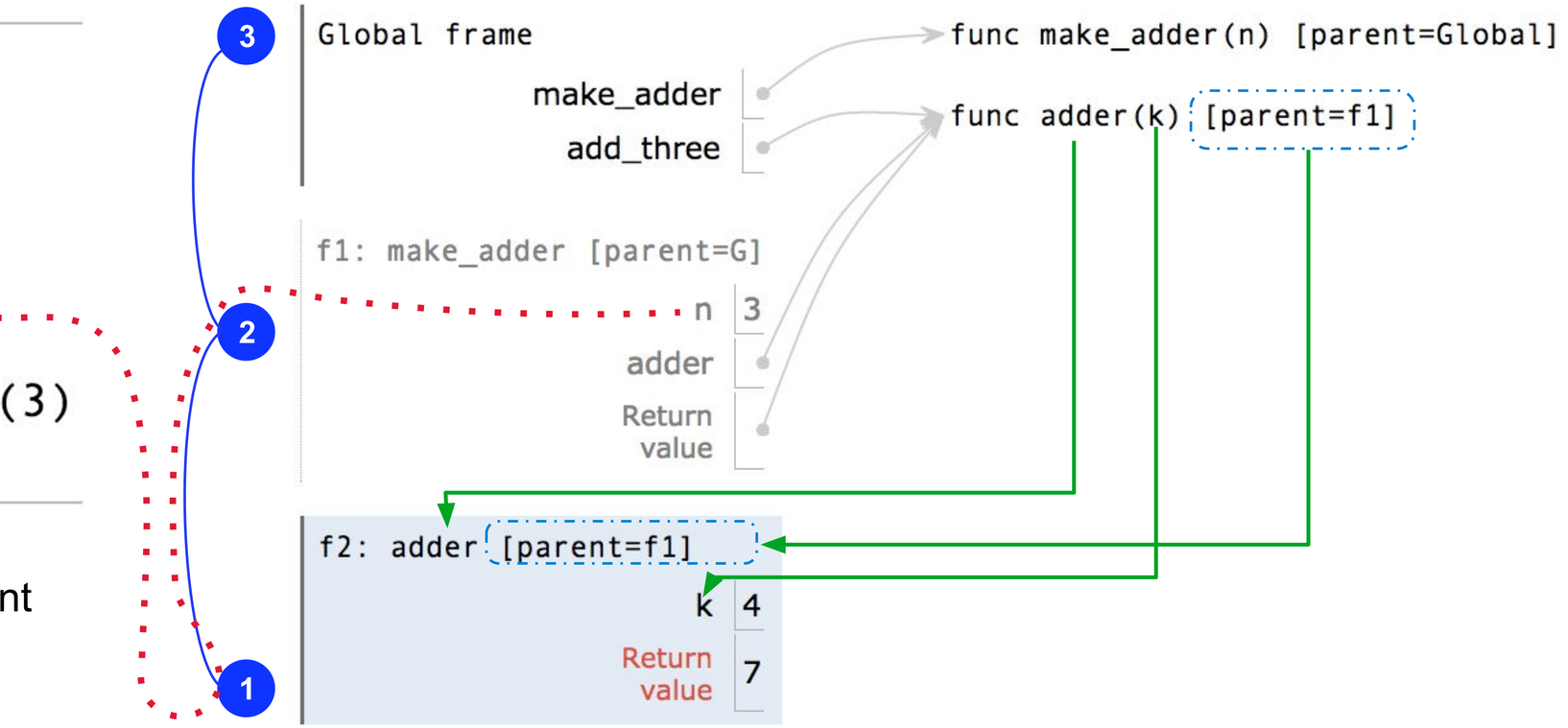
Environments for Nested Definitions

(Demo)

Environment Diagrams for Nested Def Statements

Nested def

```
1 def make_adder(n):
2     def adder(k):
3         return k + n
4     return adder
5
6 add_three = make_adder(3)
7 add_three(4)
```



- Every user-defined function has a parent frame (often global)
- The parent of a function is the frame in which it was defined
- Every local frame has a parent frame (often global)
- The parent of a frame is the parent of the function called

http://pythontutor.com/composingprograms.html#code=def%20make_adder%28n%29%3A%0A%20%20%20%20def%20adder%28k%29%3A%0A%20%20%20%20%20%20%20%20return%20k%20%2B%20n%0A%20%20%20%20return%20adder%0A%20%20%20%20%0Athree_more_than%20%3D%20make_adder%283%29%0Aresult%20%3D%20three_more_than%284%29&cumulative=false&curInstr=0&mode=display&origin=composingprograms.js&py=3&rawInputLstJSON=%5B%5D

How to Draw an Environment Diagram

When a function is defined:

Create a function value: func <name>(<formal parameters>) [parent=<label>]

Its parent is the current frame.

Bind <name> to the function value in the current frame

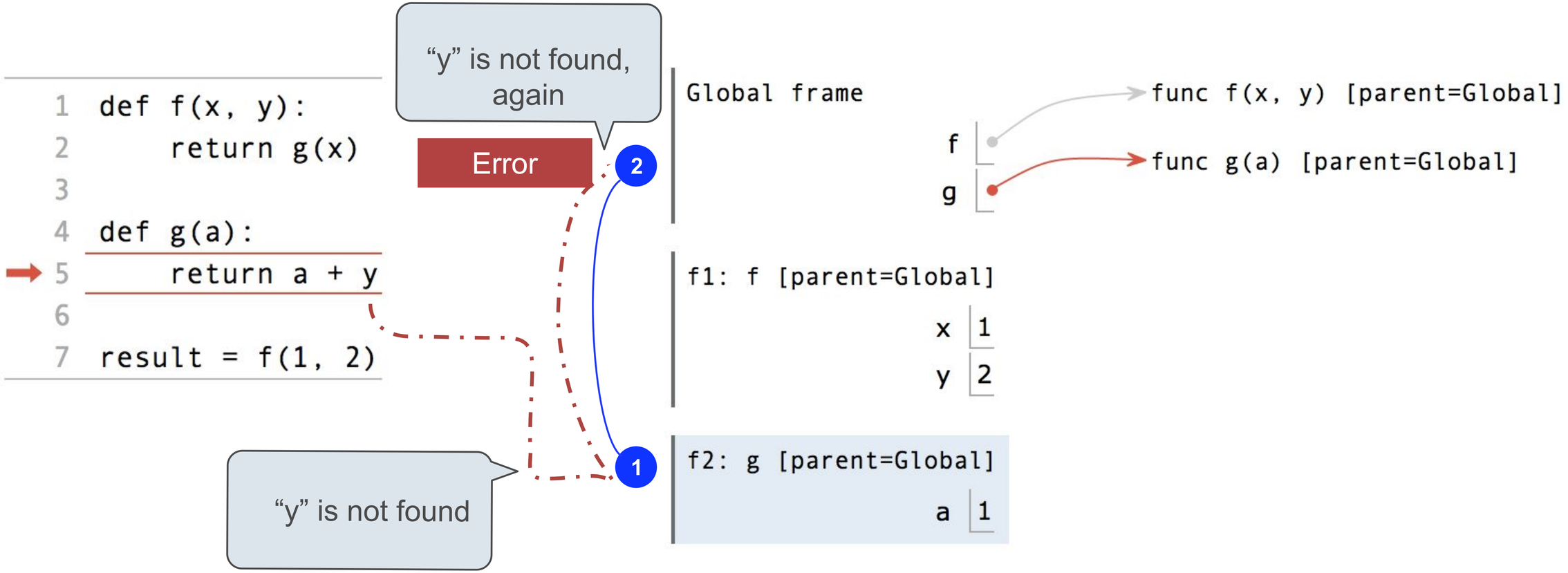
When a function is called:

1. Add a local frame, titled with the <name> of the function being called.
2. Copy the parent of the function to the local frame: [parent=<label>]
3. Bind the <formal parameters> to the arguments in the local frame.
4. Execute the body of the function in the environment that starts with the local frame.

Local Names

(Demo)

Local Names are not Visible to Other (Non-Nested) Functions



- An environment is a sequence of frames.
- The environment created by calling a top-level function (no def within def) consists of one local frame, followed by the global frame.

http://pythontutor.com/composingprograms.html#code=def%20f%28x,%20y%29%3A%0A%20%20%20%20return%20g%28x%29%0A%0Adef%20g%28a%29%3A%0A%20%20%20%20return%20a%20%2B%20y%0A%20%20%20%20%0Aresult%20%3D%20f%281,%202%29&cumulative=true&curInstr=0&mode=display&origin=composingprograms.js&py=3&rawInputLstJSON=%5B%5D

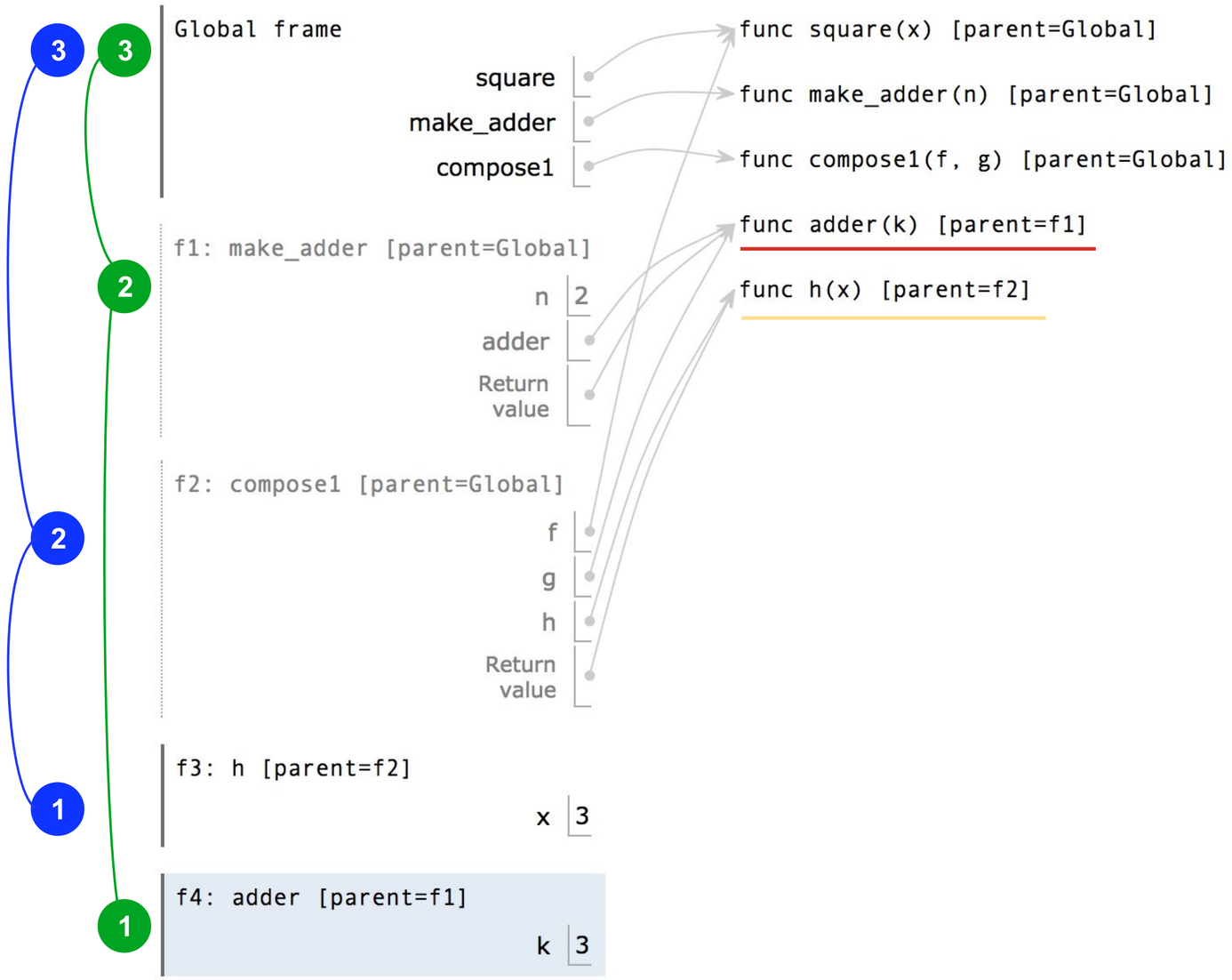
Function Composition

(Demo)

The Environment Diagram for Function Composition

```
1  def square(x):
2      return x * x
3
4  def make_adder(n):
5      def adder(k):
6          return k + n
7      return adder
8
9  def compose1(f, g):
10     def h(x):
11         return f(g(x))
12     return h
13
14 compose1(square, make_adder(2))(3)
```

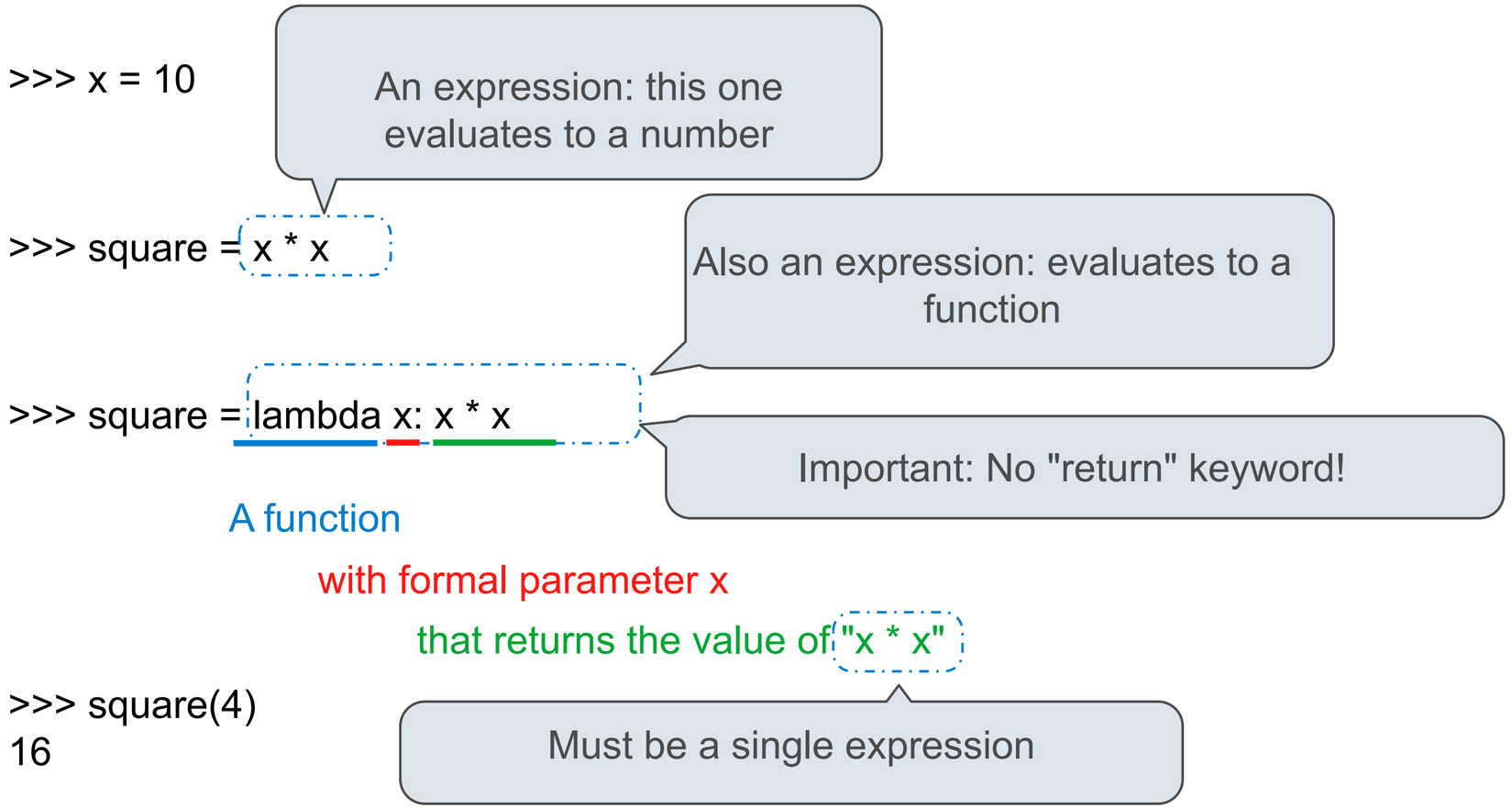
Return value of make_adder is an argument to compose1



Lambda Expressions

(Demo)

Lambda Expressions



Lambda expressions are not common in Python, but important in general

Lambda expressions in Python cannot contain statements at all!

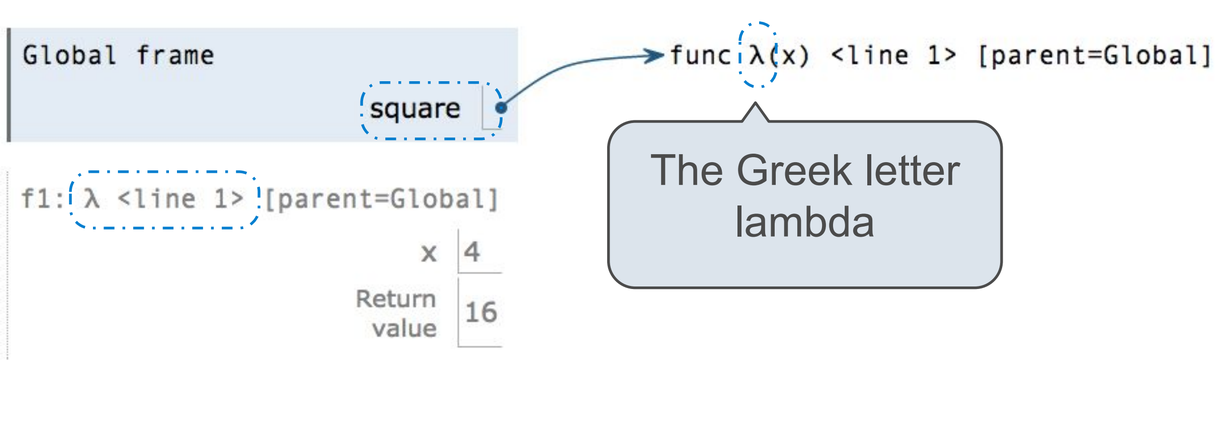
Lambda Expressions Versus Def Statements

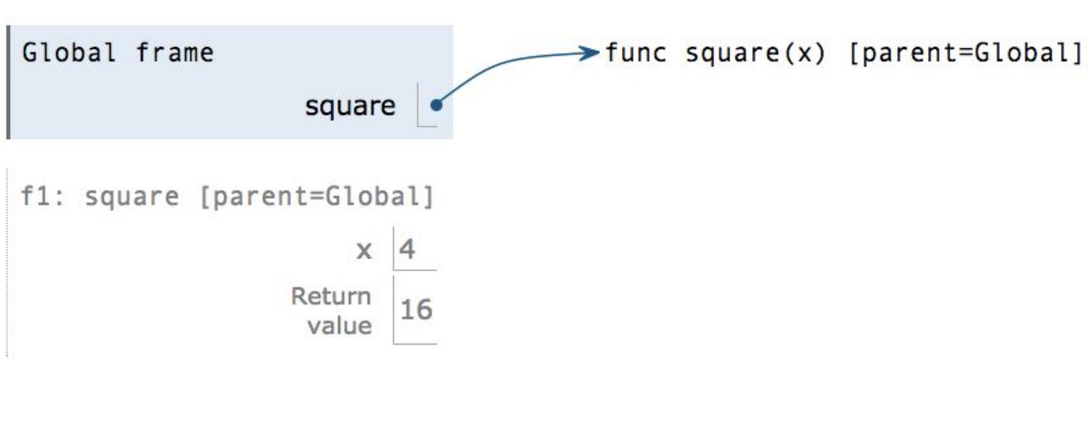
```
square = lambda x: x * x
```

V
S

```
def square(x):
    return x * x
```

- Both create a function with the same domain, range, and behavior.
- Both bind that function to the name square.
- Only the def statement gives the function an intrinsic name, which shows up in environment diagrams but doesn't affect execution (unless the function is printed).





Summary

- As we start to design functions ourselves, we want to think about giving them well-defined jobs that can apply to many situations. Functional abstraction!
- Well defined functions can help reduce redundancy in our code, which makes it more readable and adaptable
- Higher-order functions are functions that can take other functions as input, or produce other functions as output—they can help us further reduce redundancy in our code
- Functions have different behavior than control structures
- Functions can be nested within other functions
- Lambda expressions are a quick way to define simple functions within a single line