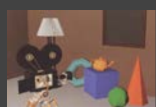
## Advanced Computer Graphics (Spring 2013)

CS 283, Lecture 18:

Basic Geometric Concepts and Rotations

Ravi Ramamoorthi

http://inst.eecs.berkeley.edu/~cs283/sp13



Most slides courtesy James O'Brien from CS294-13 Fall 2009

## Motivation

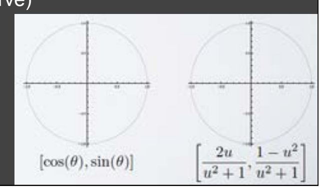
- Moving from rendering to simulation, animation
- Basic differential geometry crucial
  - How to compute frames, curvature, rotations
- This lecture relates to geometry, but focuses more on continuous concepts
- Future lectures deal with animation and simulation
- Quite mathematical, useful knowledge

## Outline

- *Parametric Curves*
- Parametric Surfaces
- Rotations in 3D

## Parametric Curves (later Surfaces)

- Curve is a geometric entity (set of points in space)
- Any local region is isomorphic to a line
- Generator function **x**(t)
  - Vector valued, or scalar function for each dimension.
  - Particular parameterization is arbitrary and not unique (not *intrinsic* to the curve)




## Arclength

$$s = A(t) = \int_0^t ||\mathbf{x}(\tau)|| \, d\tau$$

- Intrinsic parameterization of curve

$$\hat{\mathbf{x}}(s) = \mathbf{x}(A^{-1}(s))$$

  - But practical closed form may be hard to find
  - Unique up to sign change and translation

$$\frac{d\hat{\mathbf{x}}(s)}{ds} = \frac{d\mathbf{x}(t)}{dt} \left|\left| \frac{d\mathbf{x}(t)}{dt} \right|\right|^{-1} \quad \text{and} \quad \left|\left| \frac{d\hat{\mathbf{x}}(s)}{ds} \right|\right| = 1$$

## Tangents, Normals, Binormals

- Tangent vector geometric property of curve
  - Intrinsic, independent of parameterization
  - Can exist where parametric velocity is 0 or undefined

$$\mathbf{T} = \frac{d\hat{\mathbf{x}}(s)}{ds}$$

## Curvature and Normal

Note: $\mathbf{T}\cdot\mathbf{T} = 1$

$(\mathbf{T}\cdot\mathbf{T})' = (1)'$

$\mathbf{T}\cdot\mathbf{T}' = 0$

Therefore: $\mathbf{T} \perp \mathbf{T}'$

We can write: $\mathbf{T}' = \kappa\mathbf{N}$

Curvature of the curve at this point

Normal of the curve at this point

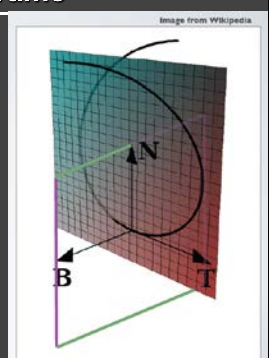
Taylor expansion implies that if curvature is zero curve must be locally a straight line.

## Frenet Frame

- Define binormal by **B** = **T** x **N**
- Gives us an orthonormal coordinate frame
  - Moves along curve
  - Gives local frame of reference
  - Not defined at inflection points where no curvature
- Can find some nice demos online

## Frenet Frame

- Osculating Plane
  - Defined by N and T
  - Locally contains curve
- Normal plane
  - Defined by N and B
  - Locally perpendicular to curve




## Torsion

$\mathbf{B}\cdot\mathbf{B} = 1 \rightarrow \mathbf{B}\cdot\mathbf{B}' = 0$

$\mathbf{B}\cdot\mathbf{T} = 0 \rightarrow \mathbf{B}'\cdot\mathbf{T} + \mathbf{B}\cdot\mathbf{T}' = 0$

$\rightarrow \mathbf{B}'\cdot\mathbf{T} = -\mathbf{B}\cdot\mathbf{T}' = -\mathbf{B}.\kappa\mathbf{N} = 0$

$\mathbf{B}' \perp \mathbf{B}$ and $\mathbf{B}' \perp \mathbf{T}$

Change in binormal is then $\mathbf{B}' = -\tau\mathbf{N}$

Torsion

If torsion is zero, we have a planar curve.

The minus sign is to make positive torsion CCW w.r.t. tangent.

## Evolution of Frenet Frame

$\mathbf{N}' \perp \mathbf{N} \rightarrow \mathbf{N}' = \alpha\mathbf{T} + \beta\mathbf{B}$

$\alpha = \mathbf{N}'\cdot\mathbf{T}$

$\beta = \mathbf{N}'\cdot\mathbf{B}$

Recall it's an orthonormal basis.

Differentiate $\mathbf{N}\cdot\mathbf{T} = 0$ and $\mathbf{N}\cdot\mathbf{B} = 0$

Yields $\mathbf{N}'\cdot\mathbf{T} = -\mathbf{N}\cdot\kappa\mathbf{N} = -\kappa$

$\mathbf{N}'\cdot\mathbf{B} = -\mathbf{N}\cdot(-\tau)\mathbf{N} = \tau$

Therefore $\mathbf{N}' = -\kappa\mathbf{T} + \tau\mathbf{B}$

We know $\mathbf{T}' = \kappa\mathbf{N}$ and $\mathbf{B}' = -\tau\mathbf{B}$

## Evolution of Frenet Frame

$\mathbf{T}' = \kappa\mathbf{N}$

$\mathbf{N}' = -\kappa\mathbf{T} + \tau\mathbf{B}$

$\mathbf{B}' = -\tau\mathbf{N}$

$$\begin{bmatrix}\mathbf{T}'\\ \mathbf{N}'\\ \mathbf{B}'\end{bmatrix} = \begin{bmatrix}0 & \kappa & 0\\ -\kappa & 0 & \tau\\ 0 & -\tau & 0\end{bmatrix}\cdot\begin{bmatrix}\mathbf{T}\\ \mathbf{N}\\ \mathbf{B}\end{bmatrix}$$

ODE for evolution of Frenet Frame

Given starting point, if you know curvature and torsion, then you can build curve.

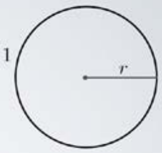
(Need "speed" also if not arclength parameterized.)

Discrete analogy: stacking up macaroni

## Radius of Curvature

$$\hat{\mathbf{x}}(s) = [r\cos\left(\frac{s}{r}\right), r\sin\left(\frac{s}{r}\right)]$$

Note that $||\hat{\mathbf{x}}'|| = 1$




$$\mathbf{T} = [-\sin\left(\frac{s}{r}\right), \cos\left(\frac{s}{r}\right)]$$

$$\mathbf{T}' = [-\frac{1}{r}\cos\left(\frac{s}{r}\right), -\frac{1}{r}\sin\left(\frac{s}{r}\right)]$$

$$\kappa = ||\mathbf{T}'|| = \frac{1}{r}$$

**Curvature is inverse of *radius of curvature*.**

## Complicated Formulae

For arclength parameterized curve

$$\kappa = ||\hat{\mathbf{x}}(s)''||$$

$$\tau = \frac{\hat{\mathbf{x}}' \cdot (\hat{\mathbf{x}}'' \times \hat{\mathbf{x}}''')}{||\hat{\mathbf{x}}''||^2}$$

For arbitrarily parameterized curve

$$\kappa = \frac{||\mathbf{x}'(t) \times \mathbf{x}''(t)||}{||\mathbf{x}'(t)||^3}$$

$$\tau = \frac{\mathbf{x}'(t) \times \mathbf{x}''(t) \cdot \mathbf{x}'''(t)}{||\mathbf{x}'(t) \times \mathbf{x}''(t)||^2}$$

## Outline

- Parametric Curves
- *Parametric Surfaces*
- Rotations in 3D

## Parametric Surfaces

- Surface is geometric entity (set of points in space)
- Any local region is isomorphic to a plane
- Generator function $\mathbf{x}(\mathbf{u})$
  - Vector valued, or scalar function for each dimension of embedding space (e.g. 2D surface embedded in 3D)
  - The parameter $\mathbf{u}$ itself is of dimension two
  - Particular parameterization is arbitrary and not unique (not *intrinsic* to the surface)

## Tangent Space

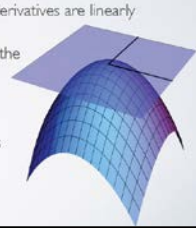
The *tangent space* at a point on a surface is the vector space spanned by

$$\frac{\partial \mathbf{x}(\mathbf{u})}{\partial u} \qquad \frac{\partial \mathbf{x}(\mathbf{u})}{\partial v}$$

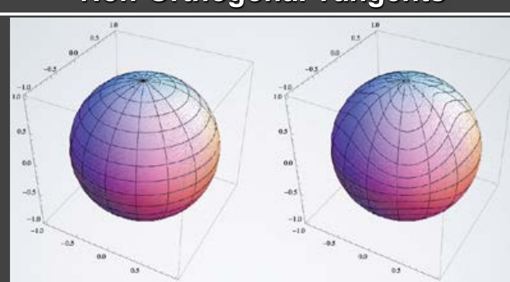
- Definition assumes that these directional derivatives are linearly independent.
- Tangent space of surface may exist even if the parameterization is bad

For surface the space is a plane

- Generalized to higher dimension manifolds



## Non-Orthogonal Tangents

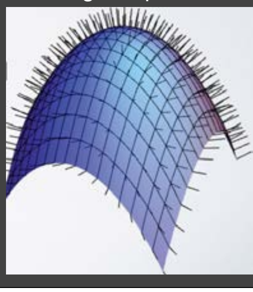


$$\begin{bmatrix} \cos(\theta 2\pi)\cos(\phi pi/2) \\ \sin(\theta 2\pi)\cos(\phi\pi/2) \\ \sin(\phi\pi/2) \end{bmatrix} \qquad \begin{bmatrix} \cos(2\pi\theta)\cos\left(\frac{1}{2}\pi\left(\frac{1}{2}(1-|\phi|)\cos(6\pi\theta)\phi+\phi\right)\right) \\ \cos\left(\frac{1}{2}\pi\left(\frac{1}{2}(1-|\phi|)\cos(6\pi\theta)\phi+\phi\right)\right)\sin(2\pi\theta) \\ \sin\left(\frac{1}{2}\pi\left(\frac{1}{2}(1-|\phi|)\cos(6\pi\theta)\phi+\phi\right)\right) \end{bmatrix}$$

$\theta \in [0..1] \quad \phi \in [-1..1]$

## Normals

- Normal at a point is unit vector perpendicular to the tangent space



$$\mathbf{N} = \frac{\partial_u \mathbf{x} \times \partial_v \mathbf{x}}{||\partial_u \mathbf{x} \times \partial_v \mathbf{x}||}$$

## First Fundamental Form

- Fundamental forms key concepts on surfaces

Pick a direction in parametric space: $\mathbf{du} = [\mathrm{d}u, \mathrm{d}v]$

Corresponding direction in the tangent plane:

$$\mathbf{dx} = \frac{\partial \mathbf{x}}{\partial u}\mathrm{d}u + \frac{\partial \mathbf{x}}{\partial v}\mathrm{d}v$$

$$\mathbf{dx} = \mathbf{du} \cdot \boldsymbol{\nabla}\mathbf{x}(\mathbf{u})$$

For unit speed in parametric space, the sped in the embedding space is

$$s^2 = \mathbf{dx} \cdot \mathbf{dx} = \mathbf{du}^{\mathsf{T}} \cdot (\boldsymbol{\nabla}\mathbf{x}) \cdot (\boldsymbol{\nabla}\mathbf{x})^{\mathsf{T}} \cdot \mathbf{du}$$

$$\mathbf{dx} \cdot \mathbf{dx} = \mathbf{du}^{\mathsf{T}} \cdot \mathbf{I} \cdot \mathbf{du}$$

$$\mathbf{I} = \begin{bmatrix} \partial_u \mathbf{x} \cdot \partial_u \mathbf{x} & \partial_v \mathbf{x} \cdot \partial_u \mathbf{x} \\ \partial_u \mathbf{x} \cdot \partial_v \mathbf{x} & \partial_v \mathbf{x} \cdot \partial_v \mathbf{x} \end{bmatrix} \qquad I_{ij} = (\partial_i x_k)(\partial_j x_k)$$

## Properties of First Fundamental Form

$$\mathbf{I} = \begin{bmatrix} \partial_u \mathbf{x} \cdot \partial_u \mathbf{x} & \partial_v \mathbf{x} \cdot \partial_u \mathbf{x} \\ \partial_u \mathbf{x} \cdot \partial_v \mathbf{x} & \partial_v \mathbf{x} \cdot \partial_v \mathbf{x} \end{bmatrix} \qquad I_{ij} = (\partial_i x_k)(\partial_j x_k)$$

- Encodes distance *metric* on surface
- For orthonormal tangents, simply identity
- Used as a metric by Green' s strain
- Invariant to translations and rotations

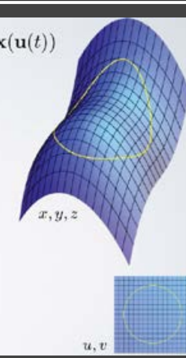
e.g.
$x'_i = R_{ij} x_j$

$$\begin{aligned} (\partial_i x'_k)(\partial_j x'_k) &= (\partial_i R_{kp} x_p)(\partial_j R_{kq} x_q) \\ &= R_{kp} R_{kq} (\partial_i x_p)(\partial_j x_q) \\ &= \delta_{pq} (\partial_i x_p)(\partial_j x_q) \\ &= (\partial_i x_p)(\partial_j x_p) \end{aligned}$$

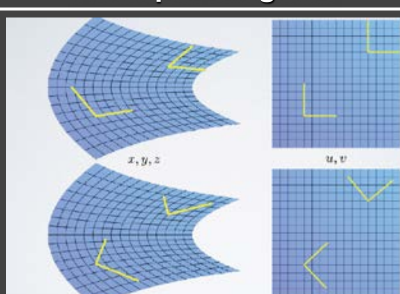
## ArcLength over Surface

$$\mathbf{c}(t) = \mathbf{x}(\mathbf{u}(t))$$

$$\begin{aligned} l &= \int_a^b \left\| \frac{\mathrm{d}\mathbf{c}(t)}{\mathrm{d}t} \right\| \mathrm{d}t \\ &= \int_a^b \sqrt{||\mathbf{dx}||^2 \mathrm{d}t} \\ &= \int_a^b \sqrt{\mathbf{dx} \cdot \mathbf{dx}}\, \mathrm{d}t \\ &= \int_a^b \sqrt{\mathbf{du}^{\mathsf{T}} \cdot \mathbf{I} \cdot \mathbf{du}}\, \mathrm{d}t \end{aligned}$$




## Principal Tangents




Bottom row is eigenvectors of $\mathbf{I}$

Not intrinsic features of the surface!

## Orthonormal Parameterization

Eigen decomposition of Fist Fundamental

$$\mathbf{I} = \mathbf{R}\mathbf{S}^2\mathbf{R}^{\mathsf{T}} = \mathbf{A}\mathbf{A}^{\mathsf{T}}$$

Define coordinate transform by

$$\mathbf{du}' = \mathbf{S}\mathbf{R}^{\mathsf{T}}\mathbf{du} = \mathbf{A}^{\mathsf{T}}\mathbf{du}$$

$$\mathbf{du} = \mathbf{R}(1/\mathbf{S})\mathbf{du}' = \mathbf{A}^{-\mathsf{T}}\mathbf{du}'$$

In transformed parameterization $\mathbf{I}$ is the identity.

$$\begin{aligned} \mathbf{du}'^{\mathsf{T}} \cdot \mathbf{I}' \cdot \mathbf{du}' &= \mathbf{du}'^{\mathsf{T}} \cdot (1/\mathbf{S}) \cdot \mathbf{R}^{\mathsf{T}} \cdot \left(\mathbf{R} \cdot \mathbf{S}^2 \cdot \mathbf{R}^{\mathsf{T}}\right) \cdot \mathbf{R} \cdot (1/\mathbf{S}) \cdot \mathbf{du}' \\ &= \mathbf{du}'^{\mathsf{T}} \cdot \left((1/\mathbf{S}) \cdot \mathbf{R}^{\mathsf{T}} \cdot \mathbf{R} \cdot \mathbf{S}^2 \cdot \mathbf{R}^{\mathsf{T}} \cdot \mathbf{R} \cdot (1/\mathbf{S}) \cdot\right) \mathbf{du}' \end{aligned}$$

Similar to definition of arclength reparameterization.

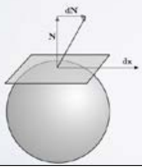
## Second Fundamental Form

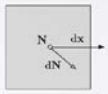
Let $\mathbf{dx}$ be some tangent direction $\quad \mathbf{dx} = \mathbf{du} \cdot \boldsymbol{\nabla}\mathbf{x}(\mathbf{u})$

The directional derivative of the normal is

$$\boldsymbol{\nabla}_{\mathbf{u}}\mathbf{N} = \frac{\partial \mathbf{N}}{\partial u} du + \frac{\partial \mathbf{N}}{\partial v} dv$$

The normal is unit length so it is perpendicular to its derivative.




As shown in top-down view, the three vectors may not be co-planar. Surface may tilt to side as point moves.

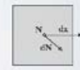
## Second Fundamental Form

Let $\mathbf{dx}$ be some tangent direction $\quad \mathbf{dx} = \mathbf{du} \cdot \boldsymbol{\nabla}\mathbf{x}(\mathbf{u})$

The directional derivative of the normal is

$$\boldsymbol{\nabla}_{\mathbf{u}}\mathbf{N} = \frac{\partial \mathbf{N}}{\partial u} du + \frac{\partial \mathbf{N}}{\partial v} dv$$

The change in normal restricted to the plane containing the tangent and normal is given by

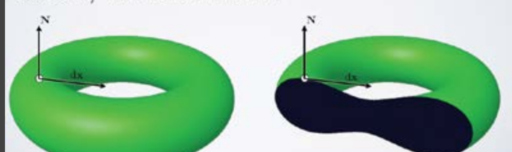


$$\begin{aligned} -\mathbf{T} \cdot \mathbf{N}_{\mathcal{T}} &= -\mathbf{dx} \cdot \boldsymbol{\nabla}_{\mathbf{u}}\mathbf{N} \\ &= -(\mathbf{du} \cdot \boldsymbol{\nabla}\mathbf{x}) \cdot (\mathbf{du} \cdot \boldsymbol{\nabla}\mathbf{N}) \\ &= \mathbf{du}^{\mathsf{T}} \begin{bmatrix} -\partial_u \mathbf{x} \cdot \partial_u \mathbf{N} & -\partial_u \mathbf{x} \cdot \partial_v \mathbf{N} \\ -\partial_v \mathbf{x} \cdot \partial_u \mathbf{N} & -\partial_v \mathbf{x} \cdot \partial_v \mathbf{N} \end{bmatrix} \mathbf{du} \end{aligned}$$

## Second Fundamental Form

$$\begin{aligned} -\mathbf{T} \cdot \mathbf{N}_{\mathcal{T}} &= \mathbf{du}^{\mathsf{T}} \begin{bmatrix} -\partial_u \mathbf{x} \cdot \partial_u \mathbf{N} & -\partial_u \mathbf{x} \cdot \partial_v \mathbf{N} \\ -\partial_v \mathbf{x} \cdot \partial_u \mathbf{N} & -\partial_v \mathbf{x} \cdot \partial_v \mathbf{N} \end{bmatrix} \mathbf{du} \\ &= \mathbf{du}^{\mathsf{T}} \mathbf{II} \mathbf{du} \end{aligned}$$

Matches definition of curvature for curve defined by cutting surface with the normal-tangent plane, but scaled by the surface metric.




## Properties

$$\begin{aligned} \mathbf{II} &= \begin{bmatrix} -\partial_u \mathbf{x} \cdot \partial_u \mathbf{N} & -\partial_u \mathbf{x} \cdot \partial_v \mathbf{N} \\ -\partial_v \mathbf{x} \cdot \partial_u \mathbf{N} & -\partial_v \mathbf{x} \cdot \partial_v \mathbf{N} \end{bmatrix} \\ &= \begin{bmatrix} \partial_{uu} \mathbf{x} \cdot \mathbf{N} & \partial_{uv} \mathbf{x} \cdot \mathbf{N} \\ \partial_{vu} \mathbf{x} \cdot \mathbf{N} & \partial_{vv} \mathbf{x} \cdot \mathbf{N} \end{bmatrix} \end{aligned}$$

Symmetry

- Easy to show second version by expanding normal
  - Box product with repeat is zero
  - Any change in normal length will be perpendicular to surface
  - Permutation of box product does not change results

## Normal Curvature

$$\kappa = \frac{\mathbf{du}^{\mathsf{T}} \cdot \mathbf{II} \cdot \mathbf{du}}{\mathbf{du}^{\mathsf{T}} \cdot \mathbf{I} \cdot \mathbf{du}}$$

Recall

$\mathbf{I} = \mathbf{R}\mathbf{S}^2\mathbf{R}^{\mathsf{T}} = \mathbf{A}\mathbf{A}^{\mathsf{T}}$

$\mathbf{du} = \mathbf{R}(1/\mathbf{S})\mathbf{du}' = \mathbf{A}^{-\mathsf{T}}\mathbf{du}'$

$$\kappa\, \mathbf{du}^{\mathsf{T}} \cdot \mathbf{I} \cdot \mathbf{du} = \mathbf{du}^{\mathsf{T}} \cdot \mathbf{II} \cdot \mathbf{du}$$

$$\kappa\, \mathbf{du}'^{\mathsf{T}} \cdot \mathbf{A}^{-1} \cdot \mathbf{I} \cdot \mathbf{A}^{-\mathsf{T}} \cdot \mathbf{du}' = \mathbf{du}'^{\mathsf{T}} \cdot \mathbf{A}^{-1} \cdot \mathbf{II} \cdot \mathbf{A}^{-\mathsf{T}} \cdot \mathbf{du}'$$

$$\kappa\, \mathbf{du}'^{\mathsf{T}} \cdot \mathbf{du}' = \mathbf{du}'^{\mathsf{T}} \cdot \mathbf{A}^{-1} \cdot \mathbf{II} \cdot \mathbf{A}^{-\mathsf{T}} \cdot \mathbf{du}'$$

$$\kappa = \frac{\mathbf{du}'^{\mathsf{T}} \cdot \mathbf{A}^{-1} \cdot \mathbf{II} \cdot \mathbf{A}^{-\mathsf{T}} \cdot \mathbf{du}'}{||\mathbf{du}'||}$$

## Principal Curvatures

$$\kappa = \frac{\mathbf{du}'^{\mathsf{T}} \cdot \mathbf{II}' \cdot \mathbf{du}'}{||\mathbf{du}'||} \qquad \mathbf{II}' = \mathbf{A}^{-1} \cdot \mathbf{II} \cdot \mathbf{A}^{-\mathsf{T}}$$

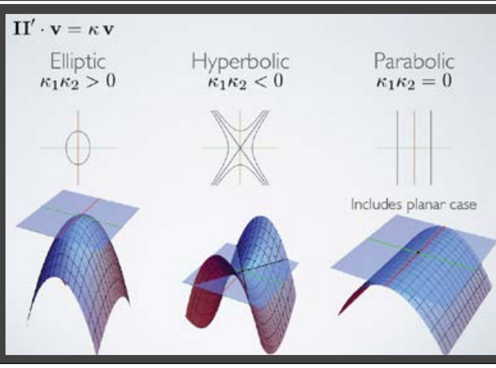
Dot product projects away "twisting" curvature

Eigenvectors are where there is nothing to project away

- Notice that it's a real and symmetric matrix

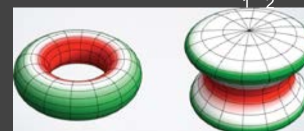
$$\mathbf{II}' \cdot \mathbf{v} = \kappa \mathbf{v}$$

## Principal Curvatures

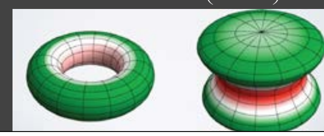



## Curvatures

- Gaussian curvature $K = \kappa_1 \kappa_2$



- Mean curvature $H = \left( \frac{\kappa_1 + \kappa_2}{2} \right)$



## Geodesic Curves

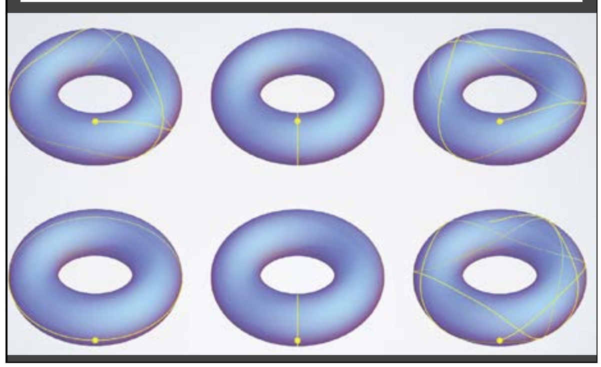
- Given a curve, $\mathbf{C}$, on a surface, $\mathbf{S}$
  - $\mathbf{C}(t) = \mathbf{S}(u(t), v(t))$
- The *geodesic curvature* is
  - $\kappa^2 = \kappa_g^2 + \kappa_n^2$
  - $\kappa_n = \kappa(\hat{\mathbf{N}}_s \cdot \hat{\mathbf{N}}_c)$
- Separates curvature into
  - What's necessary to stay on surface
  - What's wiggling in tangent plane
- Geodesics are curves with $\kappa_g = 0$
  - Generalize straight lines
  - Locally shortest path between points
  - On a circle they are great arcs

$$\frac{\mathrm{d}^2\mathbf{C}}{\mathrm{d}t^2} \cdot \frac{\partial \mathbf{S}}{\partial u_i} = 0 \quad \forall i$$

ODE for curve

$$\ddot{u}_q = (\mathbf{I}^{-1})_{qp} \frac{\partial S_k}{\partial u_p} \frac{\partial^2 S_k}{\partial u_i \partial u_j} \dot{u}_i \dot{u}_j$$

## Geodesic Curves



## Outline

- Parametric Curves
- Parametric Surfaces
- *Rotations in 3D*

## General 3D Rotations

- Non-commutative, much more complex than 2D
- General 3D axis, angle of rotation
- In axis-aligned case, simpler
  - In all cases, orthogonal matrices
  - Rows and columns of matrix are orthonormal

$$\mathbf{R}_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{bmatrix} \quad \mathbf{R}_y = \begin{bmatrix} \cos(\theta) & 0 & \sin(\theta) \\ 0 & 1 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) \end{bmatrix}$$

$$\mathbf{R}_z = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

## Arbitrary 3D rotations

Can be built from axis-aligned matrices:

$$\mathbf{R} = \mathbf{R}_{\hat{z}} \cdot \mathbf{R}_{\hat{y}} \cdot \mathbf{R}_{\hat{x}}$$

Result due to Euler... hence called

Euler Angles

Easy to store in vector

But NOT a vector:

$$\mathbf{R} = \mathrm{rot}(x, y, z)$$

## Rotation Matrices

Consider:

$$\mathbf{RI} = \begin{bmatrix} r_{xx} & r_{xy} & r_{xz} \\ r_{yx} & r_{yy} & r_{yz} \\ r_{zx} & r_{zy} & r_{zz} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Columns are coordinate axes after transformation (true for general matrices)

Rows are original axes in original system (not true for general matrices)

## Axis-Angle

Direct representation of arbitrary rotation

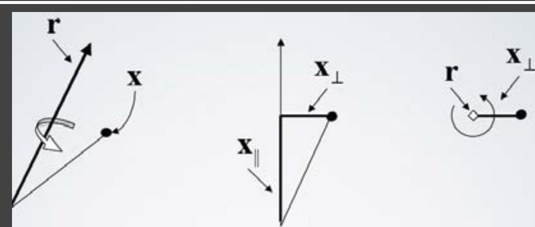
AKA: axis-angle, angular displacement vector

Rotate $\theta$ degrees about some axis

Encode $\theta$ by length of vector

$$\theta = |\mathbf{r}|$$

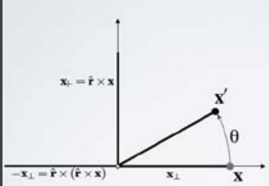

## Axis Angle split Components

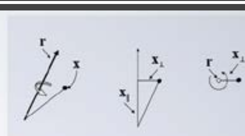



Vector expressing a point has two parts

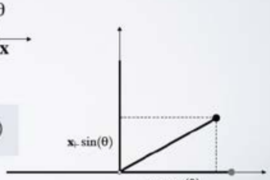
- $\mathbf{x}_\parallel$ does not change
- $\mathbf{x}_\perp$ rotates like a 2D point

## Axis Angle Components



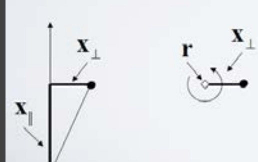



$$\mathbf{x}' = \mathbf{x}_{||} + \mathbf{x}_{\vdash} \sin(\theta) + \mathbf{x}_\perp \cos(\theta)$$




## Rodriguez Formula

$$\mathbf{x}' = \hat{\mathbf{r}}(\hat{\mathbf{r}} \cdot \mathbf{x}) + \sin(\theta)(\hat{\mathbf{r}} \times \mathbf{x}) - \cos(\theta)(\hat{\mathbf{r}} \times (\hat{\mathbf{r}} \times \mathbf{x}))$$




Linear in $\mathbf{x}$

Actually a minor variation ...

## Rodriguez Matrix

$$\mathbf{x}' = \left((\hat{\mathbf{r}}\hat{\mathbf{r}}^t) + \sin(\theta)(\hat{\mathbf{r}}\times) - \cos(\theta)(\hat{\mathbf{r}}\times)(\hat{\mathbf{r}}\times)\right)\mathbf{x}$$

$$(\hat{\mathbf{r}}\times) = \begin{bmatrix} 0 & -\hat{r}_z & \hat{r}_y \\ \hat{r}_z & 0 & -\hat{r}_x \\ -\hat{r}_y & \hat{r}_x & 0 \end{bmatrix}$$

Antisymmetric matrix

$(\mathbf{a}\times)\mathbf{b} = \mathbf{a}\times\mathbf{b}$

Easy to verify by expansion

## Exponential Maps

- Allows tumbling
- No gimbal lock
- Orientations are space within π radius ball
- Nearly unique representations
- Singularities on shells at 2 π
- Nice for interpolation

## Exponentials: Basic Properties

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$e^{i\theta} = 1 + \frac{i\theta}{1!} + \frac{-\theta^2}{2!} + \frac{-i\theta^3}{3!} + \frac{\theta^4}{4!} + \cdots$$

$$= \left(1 + \frac{-\theta^2}{2!} + \frac{\theta^4}{4!} + \cdots\right) + i\left(\frac{\theta}{1!} + \frac{-\theta^3}{3!} + \cdots\right)$$

$$= \cos(\theta) + i\sin(\theta)$$

## Matrix Exponentials

$$e^{(\hat{\mathbf{r}}\times)\theta} = \mathbf{I} + \frac{(\hat{\mathbf{r}}\times)\theta}{1!} + \frac{(\hat{\mathbf{r}}\times)^2\theta^2}{2!} + \frac{(\hat{\mathbf{r}}\times)^3\theta^3}{3!} + \frac{(\hat{\mathbf{r}}\times)^4\theta^4}{4!} + \cdots$$

But notice that: $(\hat{\mathbf{r}}\times)^3 = -(\hat{\mathbf{r}}\times)$

$$e^{(\hat{\mathbf{r}}\times)\theta} = \mathbf{I} + \frac{(\hat{\mathbf{r}}\times)\theta}{1!} + \frac{(\hat{\mathbf{r}}\times)^2\theta^2}{2!} + \frac{-(\hat{\mathbf{r}}\times)\theta^3}{3!} + \frac{-(\hat{\mathbf{r}}\times)^2\theta^4}{4!} + \cdots$$

$$e^{(\hat{\mathbf{r}}\times)\theta} = (\hat{\mathbf{r}}\times)\left(\frac{\theta}{1!} - \frac{\theta^3}{3!} + \cdots\right) + \mathbf{I} + (\hat{\mathbf{r}}\times)^2\left(+\frac{\theta^2}{2!} - \frac{\theta^4}{4!} + \cdots\right)$$

$$e^{(\hat{\mathbf{r}}\times)\theta} = (\hat{\mathbf{r}}\times)\sin(\theta) + \mathbf{I} + (\hat{\mathbf{r}}\times)^2(1 - \cos(\theta))$$

## Quaternions

- More popular than exponential maps
- Natural extension of complex numbers
- Hamilton 1843: interesting history
- Uber-complex numbers

$$\mathsf{q} = (z_1, z_2, z_3, s) = (\mathbf{z}, s)$$

$$\mathsf{q} = iz_1 + jz_2 + kz_3 + s$$

$$i^2 = j^2 = k^2 = -1 \qquad \begin{matrix} ij = k & ji = -k \\ jk = i & kj = -i \\ ki = j & ik = -j \end{matrix}$$

## Quaternion Properties

Multiplication natural consequence of defn.

$$\mathsf{q}\cdot\mathsf{p} = (\mathbf{z}_q s_p + \mathbf{z}_p s_q + \mathbf{z}_p \times \mathbf{z}_q \ , \ s_p s_q - \mathbf{z}_p \cdot \mathbf{z}_q)$$

Conjugate

$$\mathsf{q}^* = (-\mathbf{z}, s)$$

Magnitude

$$||\mathsf{q}||^2 = \mathbf{z}\cdot\mathbf{z} + s^2 = \mathsf{q}\cdot\mathsf{q}^*$$

## Quaternion Rotations

Vectors as quaternions

$$v = (\mathbf{v}, 0)$$

Rotations as quaternions

$$r = (\hat{\mathbf{r}} \sin\frac{\theta}{2}, \cos\frac{\theta}{2})$$

Rotating a vector

$$x' = r \cdot x \cdot r^*$$

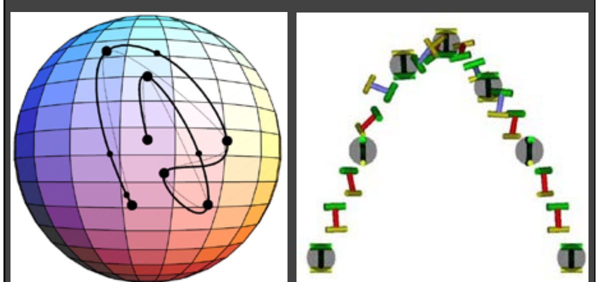
Composing rotations

$$r = r_1 \cdot r_2$$ ← Compare to Exp. Map

## Quaternions

- No tumbling
- No gimbal lock
- Orientations are double unique
- Surface of unit 3-sphere in 4D
- Nice for interpolation
  - Slerps
  - Optimal quaternion splines

## Interpolation



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