Symbolic Integration: The Stormy Decade

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Three approaches to symbolic integration in the 1960's are described. The first, from artificial intelligence, led to Slagle's SAINT and to a large degree to Moses' SIN. The second, from algebraic manipulation, led to Manove's implementation and to Horowitz' and Tobey's reexamination of the Hermite algorithm for integrating rational functions. The third, from mathematics, led to Richardson's proof of the unsolvability of the problem for a class of functions and for Risch's decision procedure for the elementary functions. Generalizations of Risch's algorithm to a class of special functions and programs for solving differential equations and for finding the definite integral are also described.

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Introduction

Symbolic integration led a stormy life in the 1960's. In the beginning of the decade only humans could determine the indefinite integral to all but the most trivial problems. The techniques used had not changed materially in 200 years. People were satisfied in considering the problem as requiring heuristic solutions and a good deal of resourcefulness and intelligence. There was no hint of the tremendous changes that were to take place in the decade to come. By the end of the decade computer programs were faster and sometimes more powerful than humans, while using techniques similar to theirs. Advances in the theory of integration yielded procedures which in a strong sense completely solved the integration problem for the usual elementary functions. The implementation of subsets of such procedures had made computers more powerful than humans or than any table for a large class of integration problems.

Three main streams of interest in symbolic integration in the 1960's can be discerned. One came from artificial intelligence and accounted for the pioneering work in Slagle's SAINT (Symbolic Automatic INTegrator) [14], and to a large degree for our own work on SIN (Symbolic INtegrator) [5]. Another came from algebraic manipulation, and accounts for Manove's rationai function integration program in the MATHLAB system [4] and also for Horowitz' [3] and Tobey's [15] theoretical analyses of that algorithm. The final stream of interest came from mathematics and accounts for Richardson's proof of the undecidability of integration for a certain class of functions [9] and Risch's decision procedures for determining the existence of an integral for elementary functions [10–13].

In this paper we shall examine symbolic integration from all three points of view. We shall first consider a symbolic integration program as a prototype of a problem solving program. Second, we shall examine the Hermite algorithm for integrating rational functions and point out computational improvements to it and remaining trouble spots in the algorithm. Then we shall introduce the ideas underlying the Risch procedure for integration which have led to the theoretical solution of the problem. Finally, we shall indicate related work on closed form solutions to ordinary differential equations and definite integration.

The Problem Solving Approach

In this section we emphasize the first viewpoint—that writing a symbolic integration program is an exercise in writing an efficient problem solving program. This can be important to other areas of algebraic manipulation, such as definite integration and power series expansions. In each case we have a well-defined problem which can be solved in a variety of ways, some of which are more efficient than others. Symbolic integration involves a second issue, namely, the form of the resulting integral. Integrals of trigonometric functions, in particular, can be written in several forms, in terms of sines and cosines, in terms of tangents of half-angles, and in terms of exponential with complex arguments. We take the view that if a user poses a problem in terms of sines and cosines, then he would prefer to see the answer in those terms, rather than in terms of complex exponentials. The reason for such a view is that one should provide the user with an answer he can comprehend easily. Thus one should not attempt to make a radical transformation of the integrand unless there is no other way to produce the integral. Integral tables appear to take this point of view also.

The design of a symbolic integration program will thus have two major goals: (1) find a solution efficiently; (2) find a solution whose form does not materially differ from that of the integrand. These goals were some of the goals of our SIN program. We shall spend most of this section in a discussion of SIN. SIN was originally written in LISP for the IBM 7094 during 1966–67. There are versions of it available for PDP-10, IBM 360, and CDC 6600 computers.

The overall strategy of SIN can be viewed as follows:

Stage 1. Attempt to solve the problem by a cheap, general method (i.e. a version of the derivative-divides method).

Stage 2. Attempt to solve the problem by one of eleven methods which are specific to a certain class of integrals (e.g. trigonometric functions, exponentials, radicals).

Stage 3. When the first two stages fail, try a general method (e.g. a heuristic integration-by-parts, the Risch algorithm).

A problem that is solved in stages 1 and 2 is solved fairly efficiently because stage 1 is passed through rather quickly, whether or not it succeeds in obtaining the integral. The algorithms in stage 2 are narrow in scope and provide efficient solutions to any integrand to which they are applicable. Furthermore, a very quick decision can be made regarding the applicability of the set of algorithms in stage 2. If both stages fail to apply to a particular integrand, then we fall through to th least stage. Here we are willing to attempt a radical transformation of the integrand in order to obtain an answer.

Some of the methods in stage 2 and the Risch algorithm are known as decision procedures. That is, they can obtain a closed form integral when such an integral exists, and they can also decide when an integral cannot exist in terms of the usual functions. Thus

$$\int e^{x^2}\, dx$$

is determined by SIN not to be integrable in closed form. Some of the methods of SIN's second stage are not decision procedures. When they fail to obtain an integral, then we still do not know whether the integral exists.

First Stage of SIN

The heart of SIN's first stage is a simple test to determine whether derivatives of a subexpression of the integrand divide the rest of the integrand. This test determines whether the integral is of the form

$$\int c \operatorname{op}\, (u(x))u'(x)\, dx,$$

where c is a constant, $u(x)$ is some function of x, $u'(x)$ is its derivative, and op is an elementary function; op can be a member of the set

{sin, cos, tan, cot, sec, csc, arcsin, arctan, arcsec, log}.

In addition, op $(u(x))$ can have the forms $u(x)$, $u(x)^{-1}$, $u(x)^d$ where $d \neq -1$, and $d^{u(x)}$, where d is a constant.

The method of solution, once the problem has been determined to possess the form above, is to search an integral table for the entry corresponding to op, and substitute $u(x)$ for each occurrence of x in the expression given in the table.

Using this method, the following examples can be integrated:

$$\int xe^{x^2}\,dx = \tfrac{1}{2}e^{x^2}, \quad \mathrm{op}(u) = d^u, \quad u(x) = x^2, \quad u'(x) = 2x, \quad c = \tfrac{1}{2};$$

$$\int 4\cos(2x+3)\,dx = 2\sin(2x+3), \mathrm{op}(u) = \cos(u), \quad u(x) = 2x+3, \quad u'(x) = 2, \quad c = 2;$$

$$\int \frac{e^x}{1+e^x}\,dx = \log(1+e^x), \quad \mathrm{op}(u) = u^{-1}, \quad u(x) = 1+e^x, \quad u'(x) = e^x, \quad c = 1;$$

$$\int \sin x\cos x\,dx = \tfrac{1}{2}\sin^2 x, \quad \mathrm{op}(u) = u, \quad u(x) = \sin(x), \quad u'(x) = \cos(x), \quad c = 1;$$

$$\int x(x^2+1)^{\frac{1}{2}}\,dx = \tfrac{1}{3}(1+x^2)^{\frac{3}{2}}, \quad \mathrm{op}(u) = u^d, \quad u(x) = x^2+1, \quad u'(x) = 2x, \quad c = \tfrac{1}{2}.$$

This method can integrate some rather trivial problems (such as the second one above), as well as some which are much less trivial such as

$$\int \cos^2(e^x)\sin(e^x)e^x\,dx = -\tfrac{1}{3}\cos^3(e^x),$$
$$op(u) = u^d, \quad u(x) = \cos(e^x), \quad u'(x) = -\sin(e^x)e^x, \quad c = -1.$$

The first stage of SIN also performs two transformations which are useful in preparing the integrand for the methods available in the later stages. The first of these transformations applies the sum rule, that is

$$\int (A_1 + A_2 + \cdots + A_n)\,dx \to \int A_1\,dx + \int A_2\,dx + \cdots + \int A_n\,dx.$$

For example,

$$\int (\sin x + e^x)\,dx$$

is transformed into

$$\int \sin x\,dx + \int e^x\,dx.$$

The latter integrals are easily obtained by the first stage.

The second transformation applies multinomial expansions to an integral which is a positive integer power of a sum. Thus

$$\int (x+e^x)^2\,dx$$

becomes

$$\int x^2\,dx + \int 2xe^x\,dx + \int e^{2x}\,dx.$$

The first and third of the resulting integrals are solved in SIN's first stage, the second in the second stage.

One of the experiments which was made with SIN was to attempt the 86 problems originally attempted by SAINT. SIN's first stage was able to solve 45 out of the 86 problems. The average time on the 7094 was 0.6 seconds. The comparable times for SAINT were, as far as can be determined, about two orders of magnitude slower, principally because SAINT was run interpretively.

Second Stage of SIN

SIN's second stage contains eleven methods which might be applicable to a given problem. A routine, called FORM, determines which of the methods should be attempted. If, for example, FORM were to encounter a subexpression $\sin(x)$ in an integrand, it would send the problem to a method which handles trigonometric functions. On the other hand, a subexpression of the form e^x would cause a call to be made to the routine which handles exponentials. In effect, FORM uses cues in the integrand to determine which methods to apply.

The eleven methods of integration are listed in Table I. Unfortunately, we can only present a condensed description of each method. The reader is referred to [5] for a more complete description. As can be seen, very few of these methods produce the integral directly. Most cause a transformation to be made which simplifies the integrand. The transformed problem is then integrated recursively by starting with stage 1. The workhorse of the eleven methods is the rational function integration routine (method 8). This routine, which we borrowed from the MATHLAB system, is described in detail below. Most of the problems solved in this stage required about 2 seconds of computation on the 7094.

Many of the patterns used by SIN's second stage [e.g. $Ax^2(c_1 + c_2x^q)^p$ in method 4] were matched by using a pattern matching language, called SCHATCHEN [5, Chap. 3]. In SCHATCHEN one can declare a pattern for a quadratic expression, such as

$$Ax^2 + Bx + C,$$

and obtain the following results:

expression	result of match		
$x^2 + 2x + 3$	$A = 1,$	$B = 2,$	$C = 3$
x^2	$A = 1,$	$B = 0,$	$C = 0$
$2x^2 + (\sqrt{2})x^2 + 3$	$A = 2+\sqrt{2},$	$B = 0,$	$C = 3$
$3 + 2x + 2x^2y$	$A = 2y,$	$B = 2,$	$C = 3$
$x^2 + 2x\sin(x)$	no match		

Third Stage of SIN

The original implementation of SIN used two different general methods in the third stage. One was a version of the integration-by-parts method (i.e.

$$\int u\,dv = uv - \int v\,du).$$

This technique uses some search to determine a good partition of the integrand. While the method is quite general, it is not very clear how to apply it in many cases.

The second method, which uses the EDGE, EDucated GuEss, heuristic, relied on concepts from the Liouville theory of integration. The EDGE heuristic generates a guess for the form of the integral based on the form of the integrand. The guess is differentiated and undetermined coefficients in it are obtained by matching the derivative with the integrand. We devised the EDGE heuristic independently of Risch's work on the Liouville theory. Risch's algorithm is clearly superior to the EDGE heuristic and later versions of SIN have used subsets of the Risch algorithm in the third stage.

Integration of Rational Functions: The Algebraic Manipulation Approach

As we mentioned earlier, the most important integration routine used by SIN is the routine for integrating rational functions (i.e. ratios of polynomials). The routine used was written by Manove for the MATHLAB system [4]. Integration of rational functions is also the main subject of the doctoral dissertations of Horowitz [3] and Tobey [15]. We shall see that the outline of the method of integrating rational functions is also used in Risch's algorithm for integrating functions containing exponentials and logarithmic terms.

The algebraic manipulation approach is one that emphasizes the need for great efficiency for relatively common operations. Rational function integration is probably the most common nontrivial integration algorithm. Though mathematicians consider the algorithm as trivial, its implementation offers considerable difficulties which have not yet been effectively surmounted. A part of the algorithm has been made more efficient by the use of special techniques in Horowitz' thesis. We shall, however, use the traditional description of the algorithm, which is very easy to explain. It should be noted that the method to be described is superior to the standard calculus text approach to rational function integration which involves solution of equations.

Suppose we are given a rational function in x, $Q(x)/S(x)$, where Q and S are relatively prime polynomials with integer coefficients. By division, we obtain the following decomposition:

$$Q(x)/S(x) = P(x) + R(x)/S(x), \text{ degree } R < \text{degree } S.$$

The polynomial part of the decomposition, $P(x)$, is trivially integrated. This will not be the case when we discuss the Risch algorithm.

We shall assume without loss of generality that R and S are relatively prime. The next step of the integration algorithm is to obtain a *square-free* decomposition of the denominator S of the form

$$S = S_1 S_2^2 S_3^3 \cdots S_k^k,$$

where each S_i and S_j are relatively prime polynomials and where each S_i has only simple roots. The polynomial S_i has as its roots the roots of S of degree i. This particular kind of factorization is easily obtained by performing greatest common divisor calculations on $S(x)$ and $S'(x)$, noting that gcd $(S, S)'$ has the same roots as S with multiplicity reduced by 1. Thus, for example, the square-free decomposition of $x^4 - x^2$ is

$$x^4 - x^2 = (x^2 - 1)(x)^2, \qquad k = 2.$$

We are in a position to perform a *partial-fraction* decomposition with relative ease. We shall indicate the first step of this decomposition. Since each pair S_i and S_j is relatively prime, S_k^k and $S_1 S_2^2 \cdots S_{k-1}^{k-1}$ will be relatively prime. Thus there exist polynomials A and B such that

$$AS_k^k + BS_1 S_2^2 \cdots S_{k-1}^{k-1} = 1.$$

A and B can be easily found by obtaining remainders as in a gcd calculation. Multiplying both sides by R we obtain

$$ARS_k^k + BRS_1 S_2^2 \cdots S_{k-1}^{k-1} = R.$$

Dividing by S we obtain

$$\frac{AR}{S_1 S_2^2 \cdots S_{k-1}^{k-1}} + \frac{BR}{S_k^k} = \frac{R}{S}.$$

We have thus performed one step in the decomposition which, when completed, will have the form

$$\frac{R}{S} = \frac{A_1(x)}{S_1} + \frac{A_2(x)}{S_2^2} + \cdots + \frac{A_k(x)}{S_k^k},$$

where degree $A_i <$ degree S_i^i, gcd $(A_i, S_i^i) = 1$. In fact, the degree of BR in x may be greater than that of S_k^k. We may, however, divide and ignore the quotient since it will be canceled by some quotient later on in the decomposition.

Thus far we have the following relationship:

$$\int \frac{R(x)}{S(x)}\,dx = \int \frac{A_1}{S_1}\,dx + \int \frac{A_2}{S_2^2}\,dx + \cdots + \int \frac{A_k}{S_k^k}\,dx.$$

For example,

$$\int \frac{1}{x^4 - x^2}\,dx = \int \frac{1}{x^2 - 1}\,dx + \int \frac{-1}{x^2}\,dx.$$

Our goal now is to obtain a reduction procedure for integrals of the form

$$\int \frac{A_i}{S_i^i}\,dx, \qquad i > 1.$$

Table I

Number, name	Solution method	Examples	Transformed into
(1) exponentials	substitute $y = c^x$, c a constant	$\int \frac{e^x}{2 + 3e^{2x}}\,dx$ $\int \frac{e^{2x}}{A + Be^{4x}}\,dx$	$\int \frac{1}{2 + 3y^2}\,dy,\ y = e^x$ $\int \frac{y}{A + By^4}\,dy,\ y = e^x$
(2) integral powers of variables	substitute $y = x^k$ where k is related to the greatest common divisor of the exponents	$\int x^3 \sin (x^2)\,dx$ $\int \frac{x^7}{x^{12} + 1}\,dx$	$\int \frac{1}{2}y \sin(y)\,dy,\ y = x^2$ $\int \frac{1}{4}\frac{y}{y^3 + 1}\,dy,\ y = x^4$
(3) rational roots of linear fractions	substitute $y = \left(\frac{ax + b}{cx + d}\right)^{1/k}$	$\int x(x+1)^{1/2}\,dx$ $\int \left(\frac{x + 1}{2x + 3}\right)^{1/2} dx$	$\int 2(y^2-1)y^2\,dy,\ y = (x+1)^{1/2}$ $\int \frac{2y^2}{(2y^2 - 1)^2}\,dy,\ y = \left(\frac{x + 1}{2x + 3}\right)^{1/2}$
(4) Chebyshev	a decision procedure for expressions of the form $Ax^r(c_1+c_2x^q)^p$, where p, q, r are rational numbers	$\int x^{1/2}(1+x)^{5/2}\,dx$ $\int x^4(1-x^2)^{-5/2}\,dx$	$\int \frac{-2y^6}{(y^2 - 1)^5}\,dy,\ y = \left(\frac{x + 1}{x}\right)^{1/2}$ $\int \frac{-1}{y^4(1 + y^2)}\,dy,\ y = \frac{(1 - x^2)^{1/2}}{x}$
(5) arctrigonometric substitutions	three arctrigonometric substitutions for integrals with expressions $(ax^2+bx+c)^{1/2}$ (the discriminant $b^2 - 4ac$ determines the substitution which will be made)	$\int \frac{x^4}{(1 - x^2)^{5/2}}\,dx$ $\int \frac{(A^2 + B^2 - B^2y^2)^{1/2}}{1 - y^2}\,dy$	$\int \frac{\sin^4 y}{\cos^4 y}\,dy,\ y = \arcsin x$ $\int \frac{1}{B}\frac{(A^2 + B^2) \cos^2 z}{[1 - (A^2 + B^2)/B^2 \sin^2 z]}\,dz$ $z = \arcsin \frac{By}{(A^2 + B^2)^{1/2}}$
(6) trigonometric functions	five classes of methods (1) integrands of the form $\sin mx \sin nx$ $\sin mx \cos nx$ $\cos mx \cos nx$, by table look-up	$\int \sin 2x \cos x\,dx$	$-\frac{1}{2} \cos x - \frac{1}{6} \cos 3x$
	(2) integrands of the form $\sin^m x \cos^n x$, by reduction of exponent	$\int \sin^2 x\,dx$	$\int (\frac{1}{2}-\frac{1}{2} \cos 2x)\,dx$
	(3) substitution of sines and cosines	$\int \frac{(A^2 + B^2 \sin^2 x)^{1/2}}{\sin x}\,dx$	$-\int \frac{[A^2 + B^2(1 - y^2)]^{1/2}}{1 - y^2}\,dy,\ y = \cos x$
	(4) substitution of tangents	$\int \frac{\sec^2 t}{1 + \sec^2 t - 3 \tan t}\,dt$	$\int \frac{1}{y^2 - 3y + 2}\,dy,\ y = \tan t$
	(5) substitution for tangent of half-angle	$\int \frac{1}{1 + \cos x}\,dx$	$\int dy,\ y = \tan \frac{1}{2}x$
(7) rational function times an exponential	decision procedure similar to Risch's	$\int \frac{x}{(x + 1)^2} e^x\,dx$ $\int \frac{2x^6 + 5x^4 + x^3 + 4x^2 + 1}{(x^2 + 1)^2} e^{x^2}\,dx$ $\int e^{x^2}\,dx$	$\frac{e^x}{x + 1}$ $\frac{2x^3 + 2x + 1}{2(x^2 + 1)} e^{x^2}$ not integrable

Table I—*Continued*

Number, name	Solution method	Examples	Transformed into
(8) rational function	Hermite method programmed by Manove	$\int \frac{x}{x^3+1}\,dx$	$-\frac{1}{3}\log(x+1) + \frac{1}{6}\log(x^2-x+1) + \frac{1}{\sqrt{3}}\arctan\left(\frac{2x-1}{\sqrt{3}}\right)$
(9) arctrigonometric or logarithmic function with rational coefficients	uses a reduction like integration-by-parts	$\int x^2 \arcsin x\,dx$ $\int x \log x\,dx$	$\frac{1}{3}x^3 \arcsin x - \int \frac{x^3}{3(1-x^2)^{1/2}}\,dx$ $\frac{1}{2}x^2 \log x - \int \frac{1}{2}x\,dx$
(10) rational functions of logarithms	attempted when method (9) is not applicable—uses a logarithmic substitution in order to reduce the problem to a form which method (7) might handle	$\int \frac{\log x}{(\log x+1)^2}\,dx$ $\int \frac{1}{\log x}\,dx$	$\int \frac{y}{(y+1)^2} e^y\,dy, \quad y = \log x$ $\int \frac{e^y}{y}\,dy, \quad y = \log x$
(11) expansion of the integrand	distributes sums over products—is applied when other methods in stage (2) fail	$\int x(\cos x+\sin x)\,dx$ $\int \left(\frac{x+e^x}{e^x}\right)dx$	$\int (x\cos x + x \sin x)\,dx$ $\int (xe^{-x}+1)\,dx$

The reduction procedure is usually credited to Hermite [2].

The polynomials S_i have only simple roots. Hence, gcd $(S_i, S_i') = 1$. Thus there exist polynomials B and C such that

$$BS_i + CS_i' = 1.$$

Therefore

$$BA_iS_i + CA_iS_i' = A_i$$

and

$$\frac{BA_i}{S_i^{i-1}} + \frac{CA_iS_i'}{S_i^i} = \frac{A_i}{S_i^i};$$

letting $BA_i = D$ and $CA_i = E$, we have

$$\int \frac{A_i}{S_i^i}\,dx = \int \frac{D}{S_i^{i-1}}\,dx + \int \frac{ES_i'}{S_i^i}\,dx.$$

By integration-by-parts we obtain

$$\int \frac{ES_i'}{S_i^i}\,dx = \frac{-E}{(i-1)S_i^{i-1}} + \int \frac{E'}{(i-1)S_i^{i-1}}\,dx.$$

Collecting the results we obtain

$$\int \frac{A_i}{S_i^i}\,dx = -\frac{E}{(i-1)S_i^{i-1}} + \int \frac{D + E'/(i-1)}{S_i^{i-1}}\,dx.$$

That is, the integral becomes a rational function plus an integral whose denominator is of lower degree in S_i. We can continue the reduction obtaining more rational terms in the integral with a final integrand whose denominator is S_i. Should the numerator of this final term be 0, then we are done and the integral is completely rational. Otherwise, we shall be forced to obtain logarithmic terms. By performing this reduction for all i, we obtain

$$\int \frac{R}{S}\,dx = \frac{T(x)}{U(x)} + \sum_{j=1}^{k} \int \frac{V_j(x)}{W_j(x)}\,dx,$$

where W_j has only simple roots, degree $V_j <$ degree W_j.

Horowitz' thesis describes an alternative routine for reaching this point in the algorithm. He is able to show that one can avoid a partial fraction decomposition and obtain the rational part of the integral more quickly by solving an appropriate system of linear equations.

The next step in the Hermite algorithm is to factor the W_j. Unfortunately, there are great practical difficulties in factoring arbitrary polynomials, even those with only simple roots. Factoring procedures such as Berlekamp's [1] will obtain only those factors having integer coefficients and thus will not be able to factor $x^4 + 1$ for all its roots. Even if one could efficiently factor all polynomials whose roots involve radicals we would still be faced with the problem that polynomials of degree 5 or greater do not have general solutions in terms of radicals. The algorithm written by Manove uses all linear and quadratic factors which could be obtained by the Kronecker factorization algorithm. In many cases, this procedure is able to obtain the complete integral. Tobey [15] points out that factorization is unavoidable unless $W_j' = c_j V_j$ in all of the terms of decomposition where c_j is a constant. In such cases the integral is simply $c_j \log W_j$. Zimmer [16] discusses algorithms for factoring over Galois extensions of the integers.

Leaving the practical aspects of the next part of the algorithm aside, we continue with the integration steps. Suppose we had a complete factorization of each W_j as follows:

$$W_j = (x - \alpha_{1j})(x - \alpha_{2j}) \cdots (x - \alpha_{kj}),$$

with the α_{ij} complex algebraic constants. Then by partial fraction expansion,

$$\int \frac{V_j}{W_j}\, dx = \int \frac{V_{1j}}{x - \alpha_{1j}}\, dx + \int \frac{V_{2j}}{x - \alpha_{2j}}\, dx + \cdots + \int \frac{V_{kj}}{x - \alpha_{kj}}\, dx.$$

The V_{ij} are, in fact, constants since their degree is less than the degree of $x - \alpha_{ij}$. Since

$$\int \frac{V_{ij}}{x - \alpha_{ij}}\, dx = V_{ij} \log (x - \alpha_{ij}),$$

the complete integral has the form

$$\int \frac{R(x)}{S(x)}\, dx = \frac{T(x)}{U(x)} + \sum_{i,j} V_{ij} \log (x - \alpha_{ij}).$$

Risch's Decision Procedure: The Approach from Mathematics

The quest for general results on integration goes back to the early nineteenth century. Laplace conjectured that the integral of an algebraic function [y is algebraic in x if there exists a nontrivial polynomial $P(x, y) = 0$, where P has integer coefficients] need contain only those algebraic functions which are present in the integrand. This conjecture was proved by Abel. Liouville examined the form of the integral of an elementary function in a series of papers in the 1830's and 1840's. Liouville's main theorem has been the basis for most of the later work in this area.

Before we can present Liouville's theorem in its modern formulation due to Risch, we shall need a few preliminary definitions. We assume that the reader has some understanding of the theory of fields. We shall assume that the field of rational functions is the ground field D in the rest of our discussion. The ground field of coefficients will be the rational numbers Q. Note that not only do the rational functions in x form a field (that is, one can add, subtract, multiply, and divide in the field), but one can also differentiate in this field. We know that one cannot integrate every rational function without requiring logarithmic extensions to the rational functions. In fact, the integral of a rational function can be represented in the following form:

$$\int R(x)\, dx = V_0(x) + \sum_{i=1}^{k} C_i \log V_i(x),$$

where $V_0 \in D$, the C_i are algebraic numbers and the V_i are in D with coefficients which are algebraic numbers. In other words, the integral of a rational function is the sum of a function in the same field with constant multiples of logarithms of functions which are also in that field. The statement of Liouville's theorem is similar to this except for modifications which allow the integral not to exist in closed form.

Liouville's theorem involves the *elementary functions*. These are obtained by making two types of extensions to the rational function field D. An *algebraic* extension of a field F is obtained from some function y such that there exists a nonlinear, irreducible polynomial $P(y) = 0$ whose coefficients are in F. For example, the square root of x can be represented by y which is the solution of $y^2 - x = 0$. A *transcendental* extension of F is obtainable from a function f which satisfies no polynomial with coefficients in F. We shall call a transcendental function a *monomial* if it is an exponential or a logarithm of a function already in the field. It is well known that e^x and $\log x$ are monomials over the rational functions. We shall be interested in only those transcendental extensions which can be formed by monomials. An elementary function is one which is in a field formed by a finite number of algebraic and monomial extensions of the rational functions.

It should be noted that not every exponential or logarithm of an element in a field F is a monomial over that field. Consider the field containing e^x and e^{x^2}, then e^{x+x^2} is algebraic over that field. Likewise, if $\log a(x)$ and $\log b(x)$ are in the field, then $\log a(x)b(x)$ is not a monomial over that field. Similarly $e^{n \log x}$ is not a monomial when n is an integer.

Now we are in a position to state Liouville's theorem. Suppose a function f is in a field of elementary functions F. Then if the integral of f is in an extension of F formed by algebraic an monomial extensions

$$\int f\,dx = V_0 + \sum_{i=1}^{k} C_i \log V_i,$$

where $V_0 \in F$, $V_i \in F$, and the C_i are constants.

The proof of Liouville's theorem is based on the differentiation properties of exponential and logarithmic monomials and of algebraic functions. The derivative of an exponential monomial always contains that exponential. If $\log^n u$ is differentiated, then the derivative contains $\log u$ except when $n = 1$. Likewise, derivatives of algebraic functions contain these algebraic functions.

For f in F to possess an integral which is an elementary function, f must possess this integral in some finite extension of F, F^* say. This integral is representable as a rational function of the monomials and algebraic functions which form F^*. By partial-fraction decomposition we can represent the potential integral as a polynomial in the monomials and algebraic functions plus some rational terms in these functions. By the differentiation properties it follows that no new functions with the exception of new logarithmic terms may arise in F^*. The logarithmic terms may only be multiplied by constants. The arguments of the new logs must also be in the original field F, for otherwise their derivative would introduce functions which are not in the integrand.

A key idea in Risch's proof of Liouville's theorem is the requirement that the monomials be algebraically independent. This allows one to perform rational operations such as partial fraction decompositions and factorization on the monomials as if they were different variables. The partial fraction decomposition of the integral allows one to obtain a canonical representation of a rational function in the monomials and algebraic functions which is extremely useful. In terms of a partial-fraction decomposition, Liouville's theorem gives the following representation of the integral, when it exists as an elementary function:

$$\int f(x, \theta_i)\,dx = P(x, \theta_i) + \sum_{j=1}^{k} \frac{R_j(x, \theta_i)}{S_j(x, \theta_i)} + \sum_{r=1}^{m} C_r \log V_r(x, \theta_i),$$

where f is in F, the θ_i are the monomials and algebraic functions in F, and P, the R_j and S_j are polynomials with the S_j square-free. The V_r are rational functions and the C_r are constants.

Risch's integration algorithm is an induction procedure on the number of monomial and algebraic extensions necessary to build up the field in which the integrand lies from the ground field of the rational functions. The integral of a rational function can be obtained in the manner already discussed. To make the induction precise for nonrational functions we have to choose an ordering of the extensions such as

e^x then $\log x$ then e^{x^2} then $\sqrt{x}$ then $\log(\sqrt{x} + 1)$. in

$$\int \frac{e^x \log x + e^{x^2}}{e^{2x} + \log(\sqrt{x} + 1)}\,dx.$$

When we examine an integrand, the monomial and algebraic extensions are frequently obvious. We must, however, be careful to allow only exponentials and logarithms which are algebraically independent of the previous monomials. This is the regularity property discussed in [6]. When we have made some choice of an ordering of the monomials and algebraic functions, then the last extension is either (a) algebraic, (b) exponential, or (c) logarithmic. The integrand which is expressible as a rational function in the monomial and algebraic extensions can be written as a sum of a polynomial part and a rational part. The integral of the rational part is easily obtained in the exponential and logarithmic cases. The integral of the polynomial part of the logarithmic case is also easily obtained. We shall describe these subsets of Risch's algorithm below.

Logarithmic Case of Risch's Algorithm

Suppose $\theta = \log u$ is the last extension used to generate F. We wish to integrate $f \in F$.

By taking a partial fraction decomposition of f we obtain

$$f(x) = A_n(x)\theta^n + A_n(x)\theta^{n-1} + \cdots + A_0(x) + \sum_{i=1}^{k} \frac{R_i(x, \theta)}{S_i^{\,i}(x, \theta)},$$

where the A_i do not contain θ, degree $R_i <$ degree $S_i^{\,i}$, and the S_i have only simple roots. Then by Liouville's theorem the integral, if it exists, has the form

$$\int f(x)\,dx = B_{n+1}(x)\theta^{n+1} + B_n(x)\theta^n + \cdots + B_0(x) + \sum_{i=1}^{k} \sum_{j=1}^{m} \frac{T_{ij}(x, \theta)}{S_i^{\,j}(x, \theta)},$$

where the B_i do not contain θ and only B_0 may contain new logarithmic extensions.

Integral of the Rational Part

The integration steps here are very similar to those in the purely rational case. We first attempt to reduce the degree of the denominator in those cases where the denominator has degree greater than 1. Since S_i has only simple roots, S_i and S_i' are relatively prime polynomials and there exist polynomials $A(x, \theta)$ and $B(x, \theta)$ (which can be found by remaindering as in a gcd operation) such that

$$AS_i + BS_i' = 1.$$

Therefore,

$$AR_iS_i + BR_iS_i' = R_i$$

and

$$\frac{AR_i}{S_i^{i-1}} + \frac{BR_iS_i'}{S_i^i} = \frac{R_i}{S_i^i}.$$

Letting $AR_i = C$ and $BR_i = D$, we have

$$\int \frac{R_i}{S_i^i}\,dx = \int \frac{D}{S_i^{i-1}}\,dx + \int \frac{ES_i'}{S_i^i}\,dx.$$

By integration by parts we obtain

$$\int \frac{R_i}{S_i^i}\,dx = \frac{-E}{(i-1)S_i^{i-1}} + \int \frac{D + E'/(i-1)}{S_i^{i-1}}\,dx.$$

Continuing in this manner we can reduce the rational part of the integral to a denominator all of whose roots are simple. The next step involves obtaining the logarithmic extension. We require that the denominator be factored. This operation is more complex than in the purely rational case since we are dealing with polynomials in several variables, the variables other than x representing the monomial and algebraic extensions. Let us ignore the practical aspects of this problem again, and continue with the integration steps. Suppose we are left with rational terms of the form R_i^*/S_i. A partial fraction decomposition would yield a decomposition such as

$$\frac{R_i^*}{S_i} = \frac{R_{i1}(x)}{\theta - a_{i1}(x)} + \frac{R_{i2}(x)}{\theta - a_{i2}(x)} + \cdots + \frac{R_{ik}(x)}{\theta - a_{ik}(x)}.$$

The integral of each one of the terms above would exist only in the case

$$R_{ij} = C(\theta - a_{ij})'.$$

The determination can be made easily by differentiation. If a constant value of C can be found, then the integral is $C \log(\theta - a_{ij})$. Otherwise the integral does not exist. Thus

$$\int \frac{1}{\log x}\,dx,$$

which already is in the form above with $\theta = \log x$, cannot be integrated in terms of elementary functions. On the other hand

$$\int \frac{1}{x \log x}\,dx,$$

which must be rewritten as

$$\int \frac{1/x}{\log x}\,dx$$

to conform with our representation, is integrable, and the integral is $\log \log x$.

The integral of the rational part of the exponential case uses the same steps except that the logarithmic terms are found in a slightly different manner.

Integral of the Polynomial Part

Suppose the polynomial part had the representation

$$A_n\theta^n + \cdots + A_0,$$

where $\theta = \log u$. Then the integral, if it exists, is a polynomial of degree $n + 1$ at most, say

$$B_{n+1}\theta^{n+1} + \cdots + B_0,$$

where the B_i $(i > 0)$ do not contain new extensions of θ.

By differentiating and comparing coefficients of powers of θ, we obtain the following:

$$0 = B_{n+1}',$$

that is B_{n+1} is a constant, say b_{n+1}, and

$$A_n = (n+1)b_{n+1}\,u'/u + B_n'.$$

Integrating both sides, we obtain

$$\int A_n(x)\,dx = (n+1)b_{n+1}\theta + B_n.$$

The integral of A_n can be found by the algorithm. The integral is less complex than the original integral since A_n does not involve θ. If the integral does not exist, the original integral does not exist either. If the integral exists, then the only new logarithmic term which may be present in it is $\log u$. Otherwise, we would violate the condition that the B_i do not involve θ $(i > 0)$. Suppose

$$\int A_n(x)\,dx = C\theta + d(x),$$

with C constant. Then

$$b_{n+1} = C/(n+1)$$

and

$$B_n(x) = d(x) + b_n,\ b_n \text{ constant}.$$

We have, in this one step, determined the constant term of the higher coefficient and the current coefficient up to a constant. Substituting $d(x) + b_n$ for $B_n(x)$ we can obtain an integral relationship for $B_{n-1}(x)$, etc. If all the B_i can be determined, the integral has been found. If some restriction on the B_i has been violated, then the integral does not exist.

Consider the following simple example:

$$\int \log x\, dx, \qquad \theta = \log x.$$

The integral, if it exists, is of the form

$$B_2\theta^2 + B_1\theta + B_0, \qquad 0 = B_2', \qquad \text{so } B_2 = b_2,$$

b_2 a constant,

$$\int 1\, dx = 2b_2\theta + B_1,$$

$$x + \text{constant} = 2b_2\theta + B_1.$$

Therefore,

$$b_2 = 0,$$

$$B_1 = x + b_1, \qquad b_1 \text{ a constant},$$

$$0 = (x + b_1)/x + B_0',$$

$$-1 = b_1/x + B_0',$$

$$\int -1\, dx = b_1\theta + B_0,$$

$$-x + \text{constant} = b_1\theta + B_0.$$

Therefore,

$$b_1 = 0,$$

$$B_0 = -x + b_0, \qquad b_0 \text{ a constant}.$$

The integral is

$$x \log x - x + \text{constant}.$$

Remaining Cases

The polynomial part of the exponential case is more complex than that of the logarithmic case because the derivative of an exponential is an exponential of the same degree. We know that the integral

$$\int A(x)\theta^n\, dx$$

must be $B(x)\theta^n$ when the integral exists and

$$\theta = e^{C(x)}.$$

Therefore

$$B' + nC'B = A.$$

This differential equation looks more complex than the original integral, but in fact it is not. The restriction on B is that it must be in the same field as A. By performing a partial-fraction decomposition of A, we can determine if such a special solution of the differential equation exists. For example, consider

$$\int e^{x^2}\, dx.$$

$A(x)$ is a polynomial of degree 0. The degree of $B(x)$ is 1, at most. Say, $B(x) = ax + b$, where a and b are constants. By differentiation we obtain

$$1 \cdot e^{x^2} = [a + 2x(ax + b)]e^{x^2}.$$

Comparing powers of x we get the following equations:

$$2a = 0, \quad 2b = 0, \quad a = 1.$$

Since this system of equations cannot be satisfied, there exists no integral in terms of elementary functions, as is well known.

The algebraic case involves very different techniques than the monomial cases. The algorithm uses knowledge about the poles of the algebraic function to yield a vector space of potential solutions by techniques of algebraic geometry. The fact that this case was shown to be decidable is very surprising since the first published conjecture that some question might be undecidable was about the integration of elliptic functions, a special case of the algebraic functions [2].

The integration algorithm is, in fact, incomplete at the present time. When one deals with exponentials and logarithms in as general a manner as Risch does then one encounters a surprising difficulty. There exists no known general algorithm for determining whether a constant involving exponentials and logarithms is 0. In [6] we mention that it is not even known whether $e + \pi$ is a rational number. Risch presents an algorithm which uses the algebraic case of the integration algorithm to determine if an exponential or logarithmic term is a monomial over a given field. The solution of the constant problem has, however, eluded him as it has everyone else.

Extensions of Risch's Algorithm

It has been known for some time that Liouville's theorem allows one to integrate functions other than those obtained through logarithmic, exponential, and algebraic extensions [8]. The logarithmic case is, in fact, easily generalized to functions obtained by integration. Consider the functions G obtained by integrating members of a field F. That is $G'(x) = f(x)$, where $f(x) \in F$. For example, the error function can be defined in this manner over the field containing the rational functions and e^{-x^2} since

$$\text{erf}'(x) = (2/\sqrt{\pi})e^{-x^2}.$$

If we assume that $G(x)$ is algebraically independent of the monomials which are in F, then we can consider the field F^* which is F extended by $G(x)$. The integral of a member of F^* will, if it exists, be a member of F^* plus constant multiples of logarithmic extensions of F^*.

In practice one is interested in finding the integral in a larger class of functions than the elementary functions. When one deals with error functions one would allow the integral to contain error functions other than

Table II

Method	Form	Procedure	Example
linear	$y' + P(x)y + Q(x) = 0$	solution is $y \exp\left(\int P\,dx\right) + \int Q\left[\exp\left(\int P\,dx\right)\right] dx = C$	$y' + y + x = 0$ becomes $ye^x + xe^x - e^x = C$
separable	$A(x)B(y)dx + C(x)D(y)dy = 0$	solution is $\int \frac{A(x)}{C(x)}\,dx + \int \frac{D(y)}{B(y)}\,dy = C$	$x(y^2 - 1)dx - y(x^2 - 1)dy = 0$ becomes $\int \frac{x}{x^2 - 1}\,dx + \int \frac{-y}{y^2 - 1}\,dy = C$
exact	$P(x, y)dx + Q(x, y)dy = 0$, where $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$	solution is $\int P\,dx + \int \left[Q - \frac{\partial}{\partial y}\left(\int P\,dx\right)\right] dy = C$ this method also contains two special cases where multipliers are generated	$(4x^3y - 12x^2y^2 + 5x^2 + 3y)y' + 6x^2y^2 - 8xy^3 + 10xy + 3y = 0$ becomes $2x^3y^2 - 4x^2y^3 + 5x^2y + 3xy = C$
Bernoulli	$f(x)y' + g(x)y + h(x)y^n = 0$ n, a constant, $n \neq 1$	substitute $u(x) = y^{1-n}$ to obtain linear differential equation	$x^2(x - 1)y' + y^2 - x(x - 2)y = 0$ becomes $y' + \frac{(x - 2)}{x(x - 1)}y + \frac{1}{x^2(x - 1)} = 0$
homogeneous	$P(x, y)dx + Q(x, y)dy = 0$ where P and Q are homogeneous functions of degree n.	substitute $u(x) = y/x$ after factoring x^n from the result one obtains a separable differential equation	$3x^2y' - 7y^2 - 3xy - x^2 = 0$ becomes $\log x - (3/\sqrt{7}) \arctan (\sqrt{7})y/x = C$
almost linear	$f(x)g(y)y' + k(x)l(y) + m(x) = 0$, where $l'(y) = g(y)$	substitute $u(y) = l(y)$ to obtain a linear differential equation	$xyy' + 2xy^2 + 1 = 0$ becomes $\frac{1}{2}xu' + 2xu + 1 = 0$, where $u(y) = y^2$
linear-coefficients	$y' + F\left(\frac{ax + by + c}{a'x + b'x + c'}\right) = 0$, where a, b, c, a', b', c' are constants, and $ab' - a'b \neq 0$	substitute $x^* = x - \frac{b'c - bc'}{a'b - ab'}$ $y^* = y - \frac{ac' - a'c}{a'b - ab'}$ to obtain a homogeneous differential equation	$(4y + 11x - 11)y' - 25y - 8x + 62 = 0$ becomes $\log (x - \frac{1}{9}) - \frac{1}{2} \log \left(1 + 2\left(\frac{y - \frac{22}{9}}{x - \frac{1}{9}}\right)\right) + \frac{3}{2} \log \left(-4 + \frac{y - \frac{22}{9}}{x - \frac{1}{9}}\right) = C$
substitution for $x^n y$	$y' + (y/x)H(x^n y) = 0$	substitute $u(x, y) = x^n y$ resulting in a separable equation	$(x - x^2y)y' - y = 0$ becomes $\frac{du}{u[1 + 1/(1 - u)]} - \frac{1}{x}\,dx = 0$

Table III

Form	Method	Example
(1) $\int_0^{2\pi} R(\sin x, \cos x)\, dx$, where R is a rational function	substitute complex exponentials for trigonometric functions resulting in a rational function $-i \int_c R\left(\frac{z^2-1}{2iz}, \frac{z^2+1}{2z}\right) \frac{dz}{z}$, where c is the unit circle-evaluate using residues	$\int_0^{2\pi} (\cos^2 x - \sin x)\, dx = \pi$
(2) $\int_{-\infty}^{\infty} F(x)\, dx$, where F is of the following types (a) rational, (b) $\frac{D(x)}{P(x)} \sin^n (mx)$, (c) $\frac{D(x)}{P(x)} \cos^n (mx)$, (d) $\frac{D(x)}{P(x)} e^{imx}$, where D is algebraic, P a polynomial, m, n real constants, and $\lim \frac{D(x)}{P(x)} = 0, \quad \text{as} \quad x \to \infty$	compute the residues either in the upper or in the lower semicircle around the complex plane	$\int_{-\infty}^{\infty} \frac{x^2 + Ax + B}{x^4 + 10x^2 + 9}\, dx = \frac{\pi B + 3\pi}{12}$ $\int_{-\infty}^{\infty} \frac{\cos x}{x^2 + a^2} = \frac{\pi e^{-a}}{a}$ $\int_{-\infty}^{\infty} \frac{\sin x}{x}\, dx = \pi$
(3) (a) $\int_{-\infty}^{\infty} F(x)\, dx$, where F is an even function, but not one of the four cases above	transform into $2 \int_0^{\infty} F(x)\, dx$ and apply method (4)	
(b) $\int_{-\infty}^{\infty} F(x)\, dx$, where F is an odd function	answer is 0	
(4) (a) $\int_0^{\infty} F(x)\, dx$, where F is rational, but has no poles at 0 and no pole of order greater than 1 on the positive real axis	integrate $\log(-z)F(z)$ around a contour in the complex z-plane cut along the positive real axis	$\int_0^{\infty} \frac{x^2 + Ax + B}{x^4 + 10x^2 + 9}\, dx = \frac{3 \log(3)A + l\pi B + 3\pi}{24}$ $\int_0^{\infty} \frac{1}{x^{1/2}(x+1)}\, dx = \pi$
(b) $\int_0^{\infty} x^{k-1} R(x)\, dx$, where R is rational, k a constant and $\lim x^k R(x) = 0, \quad \text{as } x \to 0$, $\lim x^k R(x) = 0, \quad \text{as } x \to \infty$	obtain the contour integral of $(-z)^{k-1} F(z)$	
(5) a class of integrals which result in the gamma function (a) $\int_0^{\infty} x^C e^{A x^B}\, dx$, where $\text{Re}(A) < 0$, $B > 0$, $\text{Re}(C) > -1$		$\int_0^{\infty} x^A e^{D - Bx^C}\, dx$, where $A > -1$, $B > 0$, $C > 0$ $= \frac{\Gamma((A+1)/C) e^D}{B^{(A+1)/C} C}$
(b) $\int_0^{\infty} x^{k-1}(x+1)^{-k-c}\, dx$		
(c) $\int_0^{\pi/2} \sin^n x \cos^m x\, dx$, $m > -1, \quad n > -1$		$\int_0^{2\pi} \sin^n x \cos^m x\, dx, \quad m > -1, \quad n > -1$ $= \frac{\Gamma((m+1)/2) \cdot \Gamma((n+1)/2)((-1)^{m+n} + (-1)^n + (-1)^m + 1)}{\Gamma((m+n+2)/2)}$
(d) $\int_0^1 x^n \log^k(1/x)\, dx$ $\int_0^1 x^n \log^k x\, dx, \quad n \geq 0$	these are transformed to $\int_0^{\infty}$ by the change of variable $-y = \log x$	$\int_0^1 \log^k x^2\, dx = (-1)^k \Gamma(k+1)$

those which appeared in the integrand. In effect, one wishes to consider a function G given as an integral to define a class of extensions of fields of elementary functions. The argument in Liouville's theorem can be modified to show that the integral, when it exists in terms of elementary functions and the G functions under consideration, will be in the same field as the integrand plus constant multiples of logarithmic extensions and constant multiples of G's whose arguments lie in the same field as the integrand.

The logarithmic case of Risch's algorithm can be generalized to handle such special functions. In [7] we discuss an experiment in which a table of integrals of error functions was checked by a program using this method. The program uncovered a number of errors in the table.

Solution of Ordinary Differential Equations

In [5] we describe a program called SOLDIER, SOLution of DIfferential Equations Routine, which contains eight methods for solving first-order differential equations. The methods used include most of the common techniques for solving first-order differential equations, namely the solutions for linear, separable, and exact equations. All the integration subproblems in SOLDIER use SIN to perform the integration. The program attempted 76 problems in an introductory text on differential equations and succeeded in solving 67 and in noticing a misprint in the text's solution to one problem.

SOLDIER's methods, which are indicated in Table II, are similar to SIN's first and second stage. There is, at present, no known general method for solving nonlinear ordinary differential equations. It is not at all clear that one could generalize Risch's algorithm to handle a large class of nonlinear differential equations. Thus a program for solving ordinary differential equations may have to be a collection of special methods.

There are general methods for solving nth-order linear differential equations with constant coefficients. One such method is implemented for the MATHLAB system [4]. The method uses Laplace transforms. As a result of the work on this method, it was shown that an algorithm for obtaining the inverse of a Laplace or Fourier transform of a rational function is a slight modification of the Hermite method for integrating rational functions.

Definite Integration

Wang has implemented in the MACSYMA system a number of methods for finding the definite integral of elementary functions. Since many definite integrations can be made by evaluating the indefinite integral at the limits, Wang has concentrated on cases where such a technique fails. The main method for evaluating such integrals is by residues. This involves finding an appropriate contour in the complex plane, locating the poles of the function (usually by factoring the denominator), and calculating the residues. Methods which have so far been implemented are described in Table III.

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