B.w. W11-12 → F11-12

Spin resonance
2-slit self interf.
Entangled spins

Crommie Lecture #5

10/7/03

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How do we control qubit states in Laboratory?

$|\psi(t)\rangle = \alpha(t)|0\rangle + \beta(t)|1\rangle$ how change α, β?

Turn on a field ⇒ Hamiltonian evolves the state via $e^{-i\hat{H}t/\hbar}$

$|\psi(t)\rangle = e^{-i\hat{H}t/\hbar}|\psi(0)\rangle$

For static B-field this allows us to rotate qubit state from one pt. on Bloch sphere to another via rotations:

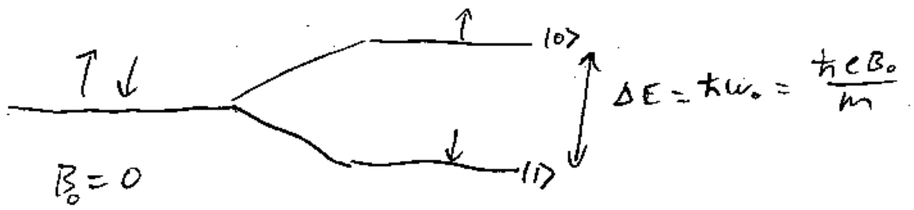
$$\hat{R}_i(\Delta\theta) = e^{-i\frac{\hat{S}_i}{\hbar}\Delta\theta}, \quad \Delta\theta = \frac{eB_0}{m}\Delta t, \quad \vec{B} = B_0\hat{x}_i$$

$i = x, y, z$

<u>Question</u>: How can we maintain energy level splitting between $|0\rangle$ and $|1\rangle$ and <u>control</u> the rate at which qubit rotates between states? [i.e., change it at a rate different from $\omega_0 = \frac{eB_0}{m}$]

<u>Answer</u>: Spin Resonance gives us a new level of control [most clearly seen in NMR]

$\boxed{\text{How it works}}$: Turn on a big dc field B_0 and a little ac field $\vec{B}_\perp \sin\omega_0 t$ that is tuned to the resonance $\omega_0 = \frac{eB_0}{m}$:



The small ac field induces controlled mixing between $|0\rangle$ and $|1\rangle$ ["SPIN FLIPS"]

Must Solve Sch. Eq'n to Understand:

$i\hbar \frac{\partial}{\partial t} |\psi(t)\rangle = \hat{H}|\psi(t)\rangle$

Convenient to Use column vector Notation: $|\psi(t)\rangle = \alpha(t)|0\rangle + \beta(t)|1\rangle = \begin{pmatrix}\alpha(t)\\ \beta(t)\end{pmatrix}$

$\Rightarrow$ What's Hamiltonian? $\hat{H} = -\vec{\mu}\cdot\vec{B} = \frac{e}{m}\vec{S}\cdot\vec{B}$

$\vec{B} = B_0\hat{z} + B_1 \cos\omega_0 t\,\hat{x}$, $\omega_0 = \frac{eB_0}{m}$

$\Rightarrow \hat{H} = \frac{e}{m}B_0\hat{S}_z + \frac{e}{m}B_1\cos\omega_0 t\,\hat{S}_x$

Now use 2x2 matrix formulation: Use Pauli Matrices $\rightarrow S_z = \frac{\hbar}{2}\sigma_3,\ S_x = \frac{\hbar}{2}\sigma$

$\Rightarrow \hat{H} = \frac{e}{m}B_0\cdot\frac{\hbar}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} + \frac{e}{m}B_1\cos\omega_0 t\cdot\frac{\hbar}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$

$\Rightarrow \hat{H} = \frac{e\hbar}{2m}\begin{pmatrix}B_0 & B_1\cos\omega_0 t\\ B_1\cos\omega_0 t & -B_0\end{pmatrix}$

Now, can Plug into Schr. Eq'n 4 Solve for $|\psi(t)\rangle$!

SKIP

But First Notice something Profound about $\hat{H}$:

$\hat{H} = \begin{pmatrix}H_{11} & H_{12}\\ H_{21} & H_{22}\end{pmatrix}$ where $H_{ij} = \langle i|\hat{H}|j\rangle$ is the matrix element connecting $|j\rangle$ to $|i\rangle$ (read right to left)

$\Rightarrow H_{12} \propto B_1$ $\quad$ $H_{12}\ (= H_{21}^*)$ is called an off-diagonal matrix element and it is special. Why?

H_{12} causes transitions between spin states!

A bit of intuition on Q.M.: If you construct a Hamiltonian matrix out of some basis, then the matrix element H_{ij} tells us how much application of the Hamiltonian tends to send a particle from state $|j\rangle$ to state $|i\rangle$. [Units are energy ⇒ rate of transitions ∝ freq. ∝ $\frac{E}{\hbar} \propto \frac{H_{ij}}{\hbar}$]

So, if we only had $\vec{B} = B_0 \hat{z}$ and $\vec{B}_1 = 0$ ⇒ what would rate of spin flip transitions be?

$$\text{rate}_{i \leftarrow j} \propto \langle i | \hat{H} | j \rangle = \langle 1 | \hat{H} | 0 \rangle = H_{21} = 0!$$

⇒ We NEED to have ⊥ $\vec{B}$ to induce "spin flips" or to mix up $|0\rangle$ and $|1\rangle$ states in $|\psi\rangle$.

This is perhaps obvious in case of spin, but not as obvious for other systems. IMPORTANT Q.M. intuition (easily lost in the math)

Now let's solve Schr. Eq'n for Spin Resonance

$$i\hbar \frac{\partial}{\partial t} |\psi(t)\rangle = \hat{H} |\psi(t)\rangle \Rightarrow i\hbar \frac{\partial}{\partial t} \begin{pmatrix} \alpha(t) \\ \beta(t) \end{pmatrix} = \frac{e\hbar}{2m} \begin{pmatrix} B_0 & B_1 \cos\omega_0 t \\ B_1 \cos\omega_0 t & -B_0 \end{pmatrix} \begin{pmatrix} \alpha(t) \\ \beta(t) \end{pmatrix}$$

⇒ Get 2 coupled differential Eq'ns:

First: Define $\omega_0 = \frac{eB_0}{m}$, $\omega_1 = \frac{eB_1}{2m}$ ← annoying factor of 2

$$\Rightarrow i \frac{\partial \alpha(t)}{\partial t} = \frac{\omega_0}{2} \alpha(t) + \omega_1 \cos\omega_0 t \, \beta(t)$$

$$i \frac{\partial \beta(t)}{\partial t} = \omega_1 \cos\omega_0 t \, \alpha(t) - \frac{\omega_0}{2} \beta(t)$$

⇒ To solve make substitution: $a(t) = \alpha(t)\, e^{i\frac{\omega_0}{2}t}$

$b(t) = \beta(t)\, e^{-i\frac{\omega_0}{2}t}$

Use approximation: $\cos\omega_0 t\; e^{i\omega_0 t} \approx 1/2$

⇒ $b(t) = \frac{2i}{\omega_1}\frac{\partial a(t)}{\partial t}$, $\frac{\partial^2 a(t)}{\partial t^2} + \frac{\omega_1^2}{4}a(t) = 0$.

⇒ $a(t) = C_1 e^{i\frac{\omega_1}{2}t} + C_2 e^{-i\frac{\omega_1}{2}t}$, $b(t) = -C_1 e^{i\frac{\omega_1}{2}t} + C_2 e^{-i\frac{\omega_1}{2}t}$

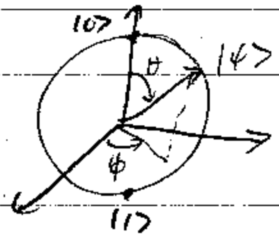
⇒ Use Boundary Conditions: Assume $t=0$ spinning up ⇒ $|\psi(0)\rangle = |0\rangle$ ⇒ $\begin{pmatrix}\alpha(0)\\ \beta(0)\end{pmatrix} = \begin{pmatrix}1\\ 0\end{pmatrix}$

⇒ Find $C_1 = C_2 = 1/2$

⇒ $a(t) = \cos\frac{\omega_1}{2}t$, $b(t) = -\sin\frac{\omega_1}{2}t$

So
$$\begin{pmatrix}\alpha(t)\\ \beta(t)\end{pmatrix} = \begin{pmatrix} e^{-i\frac{\omega_0}{2}t}\cos\frac{\omega_1}{2}t \\ -e^{+i\frac{\omega_0}{2}t}\sin\frac{\omega_1}{2}t\end{pmatrix}$$

What does this mean geometrically? Go to Bloch sphere



$|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle$

⇒ mult. by $e^{+i\frac{\omega_0}{2}}$

⇒ $|\psi(t)\rangle = e^{-i\frac{\omega_0}{2}t}\cos\frac{\omega_1}{2}t\,|0\rangle - e^{+i\frac{\omega_0}{2}t}\sin\frac{\omega_1}{2}t\,|1\rangle$

$$\Rightarrow |\psi(t)\rangle = \cos\frac{\overbrace{(\omega_1 t)}^{\theta}}{2}|0\rangle + e^{i\overbrace{(\omega_0 t + \pi)}^{\phi}}\sin\frac{\overbrace{(\omega_1 t)}^{\theta}}{2}|1\rangle$$

⇒ geometrically: $\phi = \omega_0 t + \pi$, we're spinning around $\hat{z}$ at rate $\omega_0 = \frac{eB_0}{m}$

BUT, $\theta = \omega_1 t$, so we're also crawling up the sphere at rate $\omega_1 = \frac{eB_1}{m}$

⇒ Can control ω_1 precisely by changing amplitude of B_1.

("Rabi Oscillations")

Note how this is diff. from 2 DC fields!! [just get slight rotation!]

$\Rightarrow$ Even though ω_0 is very large (GHz) $\Rightarrow$ ω_1 can be very small (kHz)

$\Rightarrow$ Can Flip spins by applying "π-pulse" : $\omega_1 \Delta t = \pi$

Note : As spins flip out of g-state $\Rightarrow$ they suck energy out of the "rf-field" ($B_1 \cos \omega_0 t$). This is easily detected and forms the basis of NMR.

Now let's talk about a little bit of quantum weirdness.

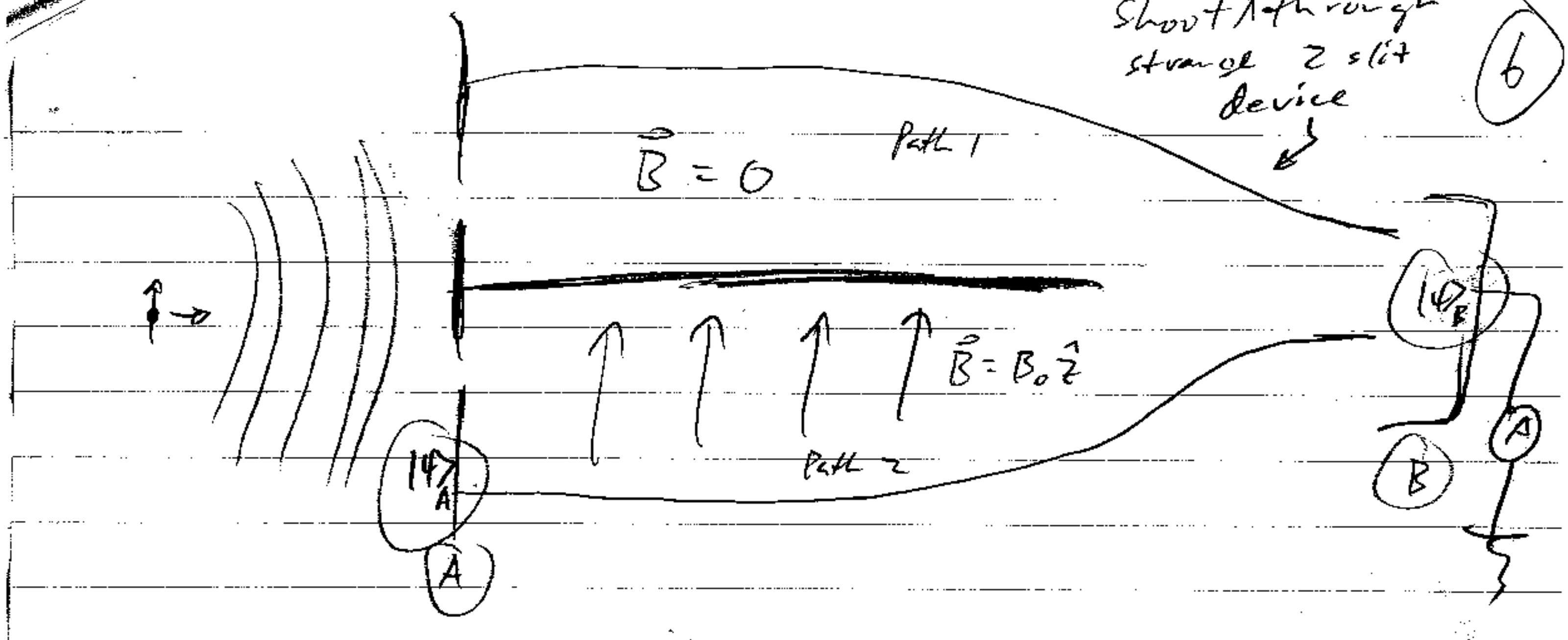
What happens if we take the spin wave fcn of a particle, break it into 2 pieces and let it interfere w/ itself?

How do this? $\rightarrow$ Use 2 slit Experiment

$\Rightarrow$ Can get Strange interference effects!

Show! $\longrightarrow$

Shoot spin through strange 2 slit device



$|\psi\rangle_A = |0\rangle$ $\Rightarrow$ Shoot spins through device, measure # spins get through to (B)

$|\psi\rangle_B = e^{-iHt/\hbar}|\psi\rangle_A = |\psi\rangle_{path\,1} + |\psi\rangle_{path\,2}$

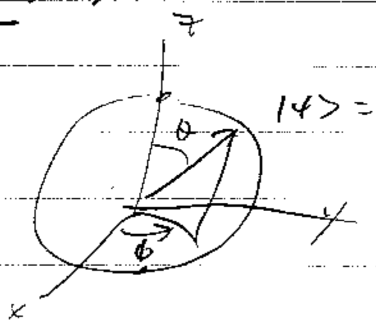
$|\psi\rangle_{path\,1} = |0\rangle$

$|\psi\rangle_{path\,2} = e^{-i\frac{\hat{S}_z}{\hbar}\Delta\phi}|0\rangle$, where $\Delta\phi = \frac{eB_0}{m}\Delta t$, $\Delta t =$ transit time

$\Rightarrow$ Suppose B_0 & Δt tuned so that $\Delta\phi = 2\pi$

$\Rightarrow$ What happens?

Naive answer:



$|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle$ WRONG

$|\psi\rangle$ rotates by 2π around $\hat{z}$ $\Rightarrow$ $\boxed{\text{nothing}}$ $|\psi\rangle_{path\,2} = |\psi\rangle_A$

Correct Answer: $|\psi\rangle_{path\,2} = e^{-i\frac{\hat{S}_z}{\hbar}\cdot 2\pi}|0\rangle = e^{-i\left(\frac{+\hbar}{2}\right)\cdot\frac{2\pi}{\hbar}}|0\rangle = e^{-i\pi}|0\rangle$

$$\boxed{|\psi\rangle_{path\,2} = -|0\rangle}$$

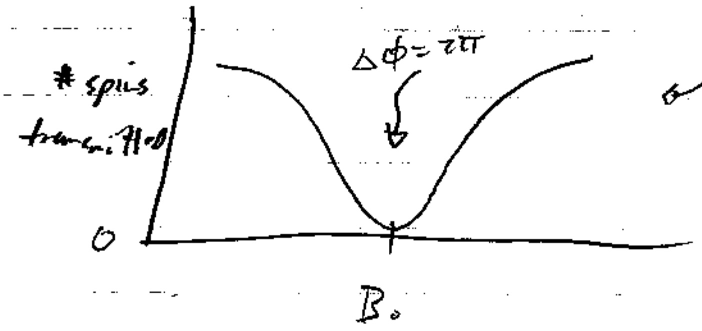
$\Rightarrow$ Normally we ignore the "−" sign since it's an overall phase factor, but Now we can't since particle interferes w/ self!!

So, what happens at B? $\Rightarrow |\psi\rangle_B = |0\rangle + -|0\rangle = 0$

$\Rightarrow$ How many particles get through? ZERO!

Can Experimentally see this!

Measure



Dip shows where Quantum interference is destructive and $|\psi\rangle \to 0$!

Strange behavior.

Has been seen

So far we've talked about 1 qubit operations
$|\psi\rangle \to |\psi'\rangle = \alpha'|0\rangle + \beta'|1\rangle$

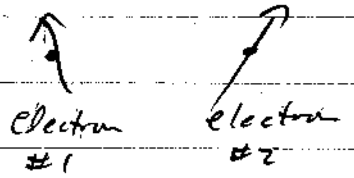
But what about entanglement? What about when there ~~are~~ is more than one qubit?

Question: How do we physically create an entangled state of 2 spins?

Answer: Must have an interaction between them!
(A 2-particle Hamiltonian!)

$\Rightarrow$ How do we create such an interaction and how does it lead to entanglement?

$\to$ Start w/ 2 electrons held next to each other:



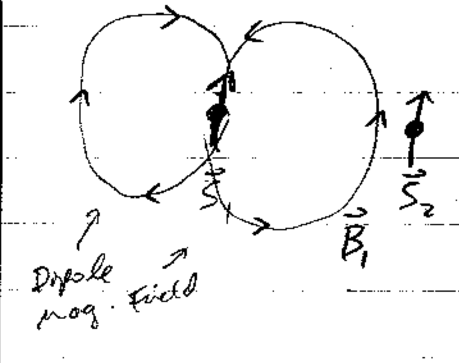
Bell state

Claim: The g-state of this system is an entangled state! $|\psi\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle_1|1\rangle_2 - |1\rangle_1|0\rangle_2\right)$

How show this? Must Solve Schr. Eq'n

$\Rightarrow$ What is $\hat{H}$? Think of the physics, how do the electrons interact? Electrons are like little current loops that can feel each other's B-field.
Think of one spin feeling $\vec{B}$-field generated by other spin

$\vec{\mu} = \frac{-e}{m}\vec{S}$, $\hat{H} = -\vec{\mu}\cdot\vec{B}$



$\Rightarrow \vec{B}_1 \propto \vec{S}_1 \Rightarrow \vec{B}_1 = A\vec{S}_1$, $A > 0$

So, $\hat{H} = -\left(\frac{-e}{m}\vec{S}_2\right)\cdot A\vec{S}_1$

$\Rightarrow \boxed{\hat{H} = C\,\vec{S}_2\cdot\vec{S}_1}$ where $C > 0$

$\Rightarrow$ What is g-state ?

Do a little trick: Consider $\vec{S}_{Total} = \vec{S}_1 + \vec{S}_2$ {A new operator!

$\Rightarrow \vec{S}_T\cdot\vec{S}_T = S_T^2 = (\vec{S}_1+\vec{S}_2)\cdot(\vec{S}_1+\vec{S}_2) = S_1^2 + S_2^2 + 2\vec{S}_2\cdot\vec{S}_1$

$\Rightarrow \vec{S}_2\cdot\vec{S}_1 = \frac{1}{2}\left(S_T^2 - S_1^2 - S_2^2\right)$

$\Rightarrow \boxed{\hat{H} = \frac{C}{2}\left(\hat{S}_T^2 - \hat{S}_1^2 - \hat{S}_2^2\right)}$ ← So, g-state is whatever state minimizes expectation value of this operator.

$\langle E\rangle = \langle\psi|\hat{H}|\psi\rangle$

Note: No matter what $|\psi\rangle$ is, $\hat{S}_1^2|\psi\rangle = \hbar^2\frac{1}{2}(\frac{1}{2}+1)|\psi\rangle$
$= \frac{3}{4}\hbar^2|\psi\rangle$

Since $|\psi\rangle = \alpha_1|0\rangle|0\rangle + \alpha_2|0\rangle|1\rangle + \alpha_3|1\rangle|0\rangle + \alpha_4|1\rangle|1\rangle$

and same for $\hat{S}_2^2$.

$\Rightarrow$ Can replace S_1^2 & S_2^2 w/ constants $\frac{3}{4}\hbar^2$

$\Rightarrow$ We see that $\hat{H} = D\hat{S}_T^2 - F$, D, F = constants > 0

$\Rightarrow$ What state $|\psi\rangle$ has smallest $\langle\psi|S_T^2|\psi\rangle$??

$\Rightarrow$ Need $|\psi\rangle$ such that $\langle\psi|S_T^2|\psi\rangle = 0$!

$\Rightarrow$ Hypothesis : $|\psi\rangle_0 = \frac{1}{\sqrt{2}}\left[|0\rangle_1|1\rangle_2 - |1\rangle_1|0\rangle_2\right]$

is the g-state! i.e., ${}_0\langle\psi|\,S_T^2\,|\psi\rangle_0 = 0$,

the minimum value.

Prove : Must calculate ${}_0\langle\psi|\,S_T^2\,|\psi_0\rangle =$

$$\frac{1}{\sqrt{2}}\left[\langle 0|\langle 1| - \langle 1|\langle 0|\right]\left[\hat{S}_1^2 + \hat{S}_2^2 + 2\,\vec{S}_2\cdot\vec{S}_1\right]\cdot\frac{1}{\sqrt{2}}\left[|0\rangle|1\rangle - |1\rangle|0\rangle\right]$$

$\Rightarrow$ Use $\vec{S}_2\cdot\vec{S}_1 = S_{2x}S_{1x} + S_{2y}S_{1y} + S_{2z}S_{1z}$ HW?

$S_{2x} \to \frac{1}{2}(S_{+2} + S_{-2})$

$S_{2y} \to \frac{1}{2i}(S_{+2} - S_{-2})$

$\Rightarrow$ Can show $\langle S_T^2\rangle_0 = 0 \Rightarrow |\psi\rangle_0$ is zero total spin state, the g-state!

So, experimentally can create Bell state by putting 2 spins next to each other, and providing a perturbation so they fall into g-state!!