Larmor Prec.
Spin Rotation
Spin Resonance

Lecture

Crommie #4

S-G / BB. Sph. connection!
S-G good for measurement, initialization of $|\psi\rangle$, not unitary transf.

Want to convert $|\psi\rangle \to |\psi'\rangle$ by rotation, not collapse of $|\psi\rangle$

Claim: We can deterministically control $|\psi(t)\rangle$ by using B-fields [$\Rightarrow$ Use to create Gates!]

$\Rightarrow$ **How?**

Must determine how spin behaves in B-field.

$\Rightarrow$ Follow UNIVERSAL QUANTUM RECIPE:

$\Rightarrow$ 1) Find $\hat{H}$ 2) Solve Schr. Eq'n!

2a) Solve (S.E.!) Time Indep. Schr. Eq'n : $\hat{H}|\psi\rangle = E_n|\psi_n\rangle$ to find E_n, $|\psi_n\rangle$

2B) Use $|\psi_n\rangle$, E_n to find $|\psi(t)\rangle$: Assume know $|\psi(0)\rangle$

$$\Rightarrow \quad |\psi(t)\rangle = e^{-i\hat{H}t/\hbar}|\psi(0)\rangle = \sum_n |\psi_n\rangle\langle\psi_n|\psi(0)\rangle\, e^{-iE_n t/\hbar}$$
[derive w/ $1 = \sum_n |\psi_n\rangle\langle\psi_n|$]

What is $\hat{H}$? Start w/ Classical $E = -\vec{\mu}\cdot\vec{B}$

$\Rightarrow$ Turn into operator: $\hat{H} = -\hat{\vec{\mu}}\cdot\vec{B}$

$\vec{B}$ is not an operator, $\hat{\vec{\mu}}$ is an operator: $\hat{\vec{\mu}} = -\frac{e}{m}\hat{\vec{S}}$ for electron

$$\Rightarrow \quad \hat{H} = \frac{e}{m}\hat{\vec{S}}\cdot\vec{B}$$
[Note: Static spin approximation, We are ignoring "orbital" component to energy, i.e., Kinetic energy, Landau Levels]

Note: Convenient thing to do is pick $\vec{B} \parallel \hat{z}$ (no loss of generality...) $\vec{B} = B_0\hat{z}$

$$\Rightarrow \quad \hat{H} = \frac{e}{m}\hat{S}_z B_0 = \frac{eB_0}{m}\hat{S}_z \quad \Rightarrow \quad \hat{H} \propto \hat{S}_z \,!$$

[Spin operators are important!!]

So, we see that Eig. states of $\hat{H}$ are eig. states of $\hat{S}_z$!

Eig. states of $\hat{H}$ are $|0\rangle$, $|1\rangle$ [since $\hat{H} \propto \hat{S}_z$!]

$\therefore$ 2 eig. states: $\hat{H}|\psi_0\rangle = E_0|\psi_0\rangle$, $\hat{H}|\psi_1\rangle = E_1|\psi_1\rangle$

where

$|\psi_0\rangle, |\psi_1\rangle = |0\rangle, |1\rangle$; But what are E_0, E_1?

Use Schr. Eq'n to find out: $\hat{H}|\psi_n\rangle = E_n|\psi_n\rangle$

Consider $|\psi_0\rangle$: $\hat{H}|\psi_0\rangle = \frac{eB_0}{m}\hat{S}_z|0\rangle = \frac{eB_0}{m}\left(+\frac{\hbar}{2}\right)|0\rangle$

So, $\boxed{E_0 = \frac{eB_0\hbar}{2m}}$

Consider $|\psi_1\rangle$: $\hat{H}|\psi_1\rangle = \frac{eB_0}{m}\hat{S}_z|1\rangle = \frac{eB_0}{m}\left(-\frac{\hbar}{2}\right)|1\rangle$

So, $\boxed{E_1 = -\frac{eB_0\hbar}{2m}}$

Which is the ground state? $|\psi_1\rangle$, has lower energy.
$\langle\vec{S}\rangle$ "wants" to anti-align w/ $\vec{B}$. $\uparrow\vec{B}$ $\downarrow\vec{S}$

So, now have eig. states, eig. energies: $|0\rangle, E_0 = \frac{eB_0\hbar}{2m}$, $|1\rangle, E_1 = \frac{-eB_0\hbar}{2m}$

$\Rightarrow$ How do we use this to find $|\psi(t)\rangle$ when $\vec{B}$ is turned on?

Easy! As long as $|\psi(t=0)\rangle$ is expressed as a sum of energy eigenstates, $|\psi_n\rangle$ $\Rightarrow$ it is easy to calculate "time evolution": $|\psi(t)\rangle = e^{-i\hat{H}t/\hbar}|\psi(t=0)\rangle$

Example: Consider one spin qubit in an arbitrary initial state $|\psi(0)\rangle = \alpha|0\rangle + \beta|1\rangle$
$\Rightarrow$ What is $|\psi\rangle$ of this qubit "t" seconds after $\vec{B} = B_0\hat{z}$ is turned on?

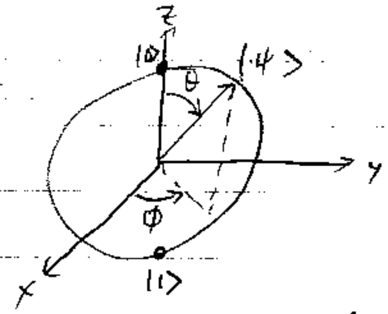
Answer: $|\psi(t)\rangle = e^{-i\hat{H}t/\hbar}|\psi(0)\rangle = e^{-i\frac{eB_0\hat{S}_z t}{m\hbar}}\left(\alpha|0\rangle + \beta|1\rangle\right)$

$\Rightarrow |\psi(t)\rangle = \alpha e^{\frac{-ieB_0t}{2m}}|0\rangle + \beta e^{\frac{+ieB_0t}{2m}}|1\rangle$ $\Rightarrow$ DONE!

O.K., great. $\boxed{\text{BUT WHAT DOES IT MEAN?!}}$

How do we interpret this result??

$\Rightarrow$ Go back to Bloch Sphere geometrical interpretation:



Remember: here convention is to multiply $|\psi\rangle$ by an overall phase factor so that α = real for $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ [ok, as long as don't change relative phase!]

$$\Rightarrow |\psi(0)\rangle = \cos\tfrac{\theta}{2}|0\rangle + \sin\tfrac{\theta}{2} e^{i\phi}|1\rangle$$

Now consider $|\psi(t)\rangle = \alpha e^{-ieB_0t/2m}|0\rangle + \beta e^{+ieB_0t/2m}|1\rangle$

$\Rightarrow$ must multiply by overall phase $e^{+ieB_0t/2m}$ to make $|0\rangle$ prefactor real.

$$\Rightarrow |\psi(t)\rangle = \alpha|0\rangle + \beta e^{+i\frac{eB_0}{m}t}|1\rangle$$

$\Rightarrow$ On Bloch sphere: $|\psi(t)\rangle = \cos\frac{\theta}{2}|0\rangle + \sin\frac{\theta}{2}e^{i\phi}\cdot e^{i\frac{eB_0}{m}t}|1\rangle$

$\Rightarrow$ Define $\omega_0 = \frac{eB}{m}$ $\Rightarrow$ $\boxed{|\psi(t)\rangle = \cos\frac{\theta}{2}|0\rangle + \sin\frac{\theta}{2}e^{i(\phi+\omega_0 t)}|1\rangle}$

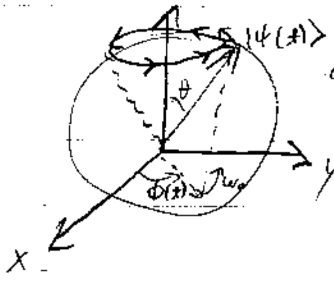
What is Geometrical Interpretation?

$|\psi(t)\rangle$ is spinning around $\hat{z}$-axis w/ freq. ω_0!

θ is unchanged, but ϕ is winding at rate ω_0!

$\Rightarrow$ Spin vector rotates around applied $\vec{B}$-field at rate $\omega_0 = \frac{eB}{m}$!

$\boxed{\text{LARMOR PRECESSION}}$



So, if $\vec{B} = B_0\hat{z}$ is turned on for time = "t" $\Rightarrow$ qubit rotates about $\vec{B}$ by an angle $\Delta\phi = \omega_0 t$!!

Turn $\vec{B} = B_0 \hat{z}$ on for "t" seconds $\Rightarrow$

In the language of operators we would say that
$|\psi(t)\rangle = e^{-i\hat{H}t/\hbar}|\psi(0)\rangle = e^{-i\frac{eB_0 S_z t}{m\hbar}}|\psi(0)\rangle = e^{-i\frac{\hat{S}_z}{\hbar}(\omega_0 t)}|\psi(0)\rangle$

$\Rightarrow$ The operator $e^{-i\frac{\hat{S}_z}{\hbar}(\omega_0 t)}$ ROTATES $|\psi(0)\rangle$ about $\hat{z}$ by $\Delta\phi = \omega_0 t$

$\Rightarrow$ Define new operator $\hat{R}_z(\Delta\phi) = e^{-i\frac{\hat{S}_z}{\hbar}\Delta\phi}$

This operator rotates a qubit state on the Bloch sphere by an angle $\Delta\phi$ about the z-axis!
Is this a gate? Yes!

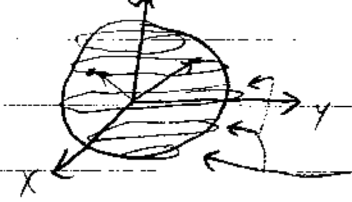
Is it Unitary? Check $R^\dagger R = ?$ $R^\dagger(\Delta\phi) = e^{+i\frac{\hat{S}_z^\dagger}{\hbar}\Delta\phi} = e^{+i\frac{\hat{S}_z}{\hbar}\Delta\phi}$
$\Rightarrow R^\dagger R = e^{+i\frac{\hat{S}_z}{\hbar}\Delta\phi} \cdot e^{-i\frac{\hat{S}_z}{\hbar}\Delta\phi} = 1 \Rightarrow$ YES

How do we create $\hat{R}_z(\Delta\phi)$ in the laboratory?

Simply turn on $\vec{B} \parallel \hat{z}$ for time Δt such that $\omega_0 \Delta t = \Delta\phi$
where $\omega_0 = \frac{eB}{m}$.

Now we have a way of reversibly transforming a single qubit from state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$
to new state $|\psi'\rangle = \alpha'|0\rangle + \beta'|1\rangle$

But there's a catch: The locus of possible α', β' is limited to points of constant LATITUDE on the Bloch sphere.

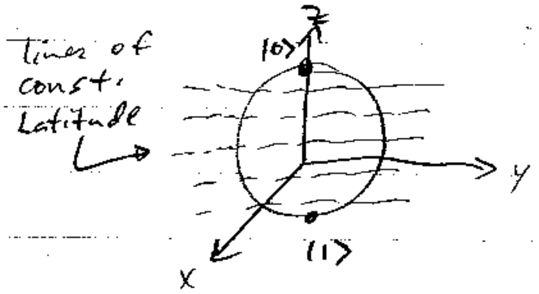


If $\vec{B} \parallel \hat{z} \Rightarrow$ we can only transform states from one pt. to another at constant latitude on Bloch sphere.

This is a severe limitation!

Question: How do we experimentally transform a qubit state from a pt. on Bloch sphere at one latitude to a point having a different Latitude?

Example: How do you transform a qubit in the state $|\psi_1\rangle = |0\rangle$ to the state $|\psi_2\rangle = |1\rangle$??



$|0\rangle$ & $|1\rangle$ are at completely different latitudes on Bloch sphere.
$\vec{B} \parallel \hat{z}$ will NOT HELP US here.

What do we do?

Answer: Turn on $\vec{B}$-field pointing in Another Direction!!

Suppose we turn on $\vec{B} \parallel \hat{y}$ for time "t": $\vec{B} = B_0 \hat{y}$ $\Rightarrow$ What happens?

By symmetry, the Physics must be the same as when we pt. $\vec{B} \parallel \hat{z}$, since there's no intrinsic difference between $\hat{y}$ & $\hat{z}$. $\Rightarrow \therefore$ we expect $|\psi\rangle$ to rotate about $\hat{y}$ with frequency $\omega_0 = \frac{eB_0}{m}$!!

Using operators $\Rightarrow$ $\hat{H} = \frac{e}{m}\vec{S}\cdot\vec{B} = \frac{e}{m}B_0\hat{S}_y$

$\Rightarrow$ Rotated $|\psi_2\rangle = e^{-i\hat{H}t/\hbar}|\psi_1\rangle = e^{-i\frac{e}{m}B_0\frac{\hat{S}_y}{\hbar}t}|0\rangle$ for $|\psi_1\rangle = |0\rangle$

$$\Rightarrow |\psi_2\rangle = e^{-i\frac{\hat{S}_y}{\hbar}\Delta\theta}|0\rangle, \quad \text{where } \Delta\theta = \omega_0 t,\ \omega_0 = \frac{eB_0}{m}$$

$\Rightarrow$ To transform $|0\rangle$ into $|1\rangle$ we need $\Delta\theta = \pi$

$\Rightarrow$ Show that this works!! (H.w.)

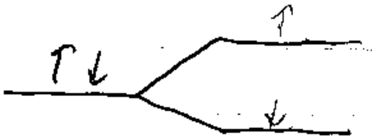
In general: Can transform a spin qubit from one state on the Bloch sphere to any other state through a series of rotations $\hat{R}_x(\gamma)$, $\hat{R}_y(\theta)$, $\hat{R}_z(\phi)$

where
$$\hat{R}_x(\gamma) = e^{-i\frac{\hat{S}_x}{\hbar}\Delta\gamma}, \quad \Delta\gamma = \frac{eB_0}{m}t, \quad \vec{B} = B_0\hat{x}$$
$$\hat{R}_y(\theta) = e^{-i\frac{\hat{S}_y}{\hbar}\Delta\theta}, \quad \Delta\theta = \frac{eB_0}{m}t, \quad \vec{B} = B_0\hat{y}$$
$$\hat{R}_z(\phi) = e^{-i\frac{\hat{S}_z}{\hbar}\Delta\phi}, \quad \Delta\phi = \frac{eB_0}{m}t, \quad \vec{B} = B_0\hat{z}$$

But, in real life there is a trick that people use:

Spin Resonance

Suppose you have a bunch of randomly aligned spins and you want to align them. ⇒ What can you do? Turn on Big $\vec{B}$-field, $\vec{B} = B_0\hat{z}$

⇒ 

$\Delta E = \hbar\omega_0, \ \omega_0 = \frac{eB_0}{m}$

Spins will all tend to line up spin down: $|\psi\rangle = |1\rangle$
Now, suppose we want to keep the big field, $\vec{B} = B_0\hat{z}$, on so that spins don't randomly mis-align, but we want to controllably flip spins from state $|\psi\rangle = |1\rangle$ to state $|\psi\rangle = |0\rangle$. Suppose, also, that we want to control the rate at which the spin states rotate.

One way is to turn $\vec{B}$ off, realign it and turn it back on. But then, if you want slow transition, you need small static $\vec{B}$-fields, and this is hard to control.

Another way: $\boxed{\text{Spin Resonance}}$

The Spin Resonance Trick: Turn on a big $\vec{B}$-field aligned along $\hat{z}$: $\vec{B}_0 = B_0 \hat{z}$ $\Rightarrow$ Turn on a smaller $\vec{B}$-field aligned $\perp$ to $\vec{B}_0$ that oscillates at $\omega_0 = \frac{eB_0}{m}$. This gives you exquisite control of the resulting spin state!!

How? Must solve Schr. Eq'n to Understand:

1) Find $\hat{H}$ $\Rightarrow$ 2) Solve T.D.S.E.: $i\hbar \frac{\partial}{\partial t}|\psi(t)\rangle = \hat{H}|\psi(t)\rangle$

Set up: Assume $\vec{B}_0 \parallel \hat{z}$ $\Rightarrow$ turn on small field $\vec{B}_1 = B_1 \cos\omega_0 t \, \hat{x}$

$\Rightarrow$ Consider general state $|\psi(t)\rangle = \alpha(t)|0\rangle + \beta(t)|1\rangle = \begin{pmatrix} \alpha(t) \\ \beta(t) \end{pmatrix}$

Want to find how $\alpha(t)$, $\beta(t)$ behave in response to $\vec{B} = B_0\hat{z} + B_1 \cos\omega_0 t \, \hat{x}$, $\omega_0 = \frac{eB_0}{m}$

$\Rightarrow$ TDSE: $i\hbar \frac{\partial}{\partial t}\begin{pmatrix} \alpha(t) \\ \beta(t) \end{pmatrix} = \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix}\begin{pmatrix} \alpha(t) \\ \beta(t) \end{pmatrix}$

What are H_{11}, H_{12}, etc.? Use $|0\rangle$, $|1\rangle$ basis ($\hat{S}_z$ eg. states)
$\#1$ $\#2$

$\cancel{\Rightarrow \text{Find } H_{11} = \langle 0|\hat{H}|0\rangle}$

What is $\hat{H}$? $\hat{H} = \frac{e}{m}\vec{S}\cdot\vec{B}$ just like before! Only now has t-Dependence!

$\Rightarrow \hat{H} = \frac{e}{m}\vec{S}\cdot\left(B_0\hat{z} + B_1\cos\omega_0 t\,\hat{x}\right) = \frac{e}{m}B_0\hat{S}_z + \frac{e}{m}B_1\cos\omega_0 t\,\hat{S}_x$

Use Pauli Matrices!!

$\cancel{\Rightarrow H_{11} = \langle 0|\left(\frac{e}{m}B_0\hat{S}_z + \frac{e}{m}B_1\cos\omega_0 t\,\hat{S}_x\right)|0\rangle}$ $\cancel{\text{Use } \hat{S}_x = \frac{1}{2}(\hat{S}_+ + \hat{S}_-)}$