1 Readings

Benenti, Casati, and Strini:

Superdense Coding Ch.4.4

No Cloning Ch.4.2

2 Superdense Coding

Super dense coding is the less popular sibling of teleportation. It can actually be viewed as the process in reverse. The idea is to transmit two classical bits of information by only sending one quantum bit. The process starts out with an EPR pair that is shared between the sender (Alice) and receiver (Bob).

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

The first qubit is Alice's, the second is Bob's.

Alice wants to send a two bit message to Bob, e.g., 00, 01, 10, or 11. She first performs a single qubit operation on her qubit which transforms the EPR pair according to which message she wants to send:

- for a message **00**: Alice applies I (i.e., does nothing)
- for a message **01**: Alice applies X
- for a message **10**: Alice applies Z
- for a message **11**: Alice applies iY (i.e. both X and Z)

This transforms the EPR pair $|\phi^+\rangle$ into the four Bell (EPR) states $|\phi^+\rangle, |\phi^-\rangle, |\psi^+\rangle$, and $|\psi^-\rangle$, respectively:

- **00**: $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \longrightarrow \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$
- **01**: $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \longrightarrow \frac{1}{\sqrt{2}}(|10\rangle + |01\rangle) = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$
- **10**: $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \longrightarrow \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle) = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}$

- **11**: $i\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right) \longrightarrow \frac{1}{\sqrt{2}}\left(|01\rangle - |10\rangle\right) = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}$

The key to super-dense coding is that these four Bell states are orthonormal and are hence distinguishable by a quantum measurement, in this case a two-qubit measurement made by Bob. This measurement can be performed directly in the Bell basis. Alternatively, Bob can first run the quantum circuit for forming Bell states from computational basis states backwards, and then measure in the computational basis to get either 00, 01, 10, or 11. See Fig.2.

Examining this process in more detail:

$|\psi_{00}\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right) = \frac{1}{\sqrt{2}}\left(|0\rangle|0\rangle + |1\rangle|1\rangle\right)$

apply $CNot$ gives us: $\frac{1}{\sqrt{2}}\left(|0\rangle|0\rangle + |1\rangle|0\rangle\right)$

applying H on 1st qubit gives us: $\frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}\left(\left(|0\rangle + |1\rangle\right)|0\rangle + \left(|0\rangle - |1\rangle\right)|0\rangle\right) =$

$\frac{1}{2}\left(|00\rangle + |10\rangle + |00\rangle - |10\rangle\right) = |00\rangle$

Bob now measures both qubits to get Alice's message 00

$|\psi_{01}\rangle = \frac{1}{\sqrt{2}}\left(|10\rangle + |01\rangle\right) = \frac{1}{\sqrt{2}}\left(|1\rangle|0\rangle + |0\rangle|1\rangle\right)$

apply $CNot$ gives us: $\frac{1}{\sqrt{2}}\left(|1\rangle|1\rangle + |0\rangle|1\rangle\right)$

applying H on 1st qubit gives us: $\frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}\left(\left(|0\rangle - |1\rangle\right)|1\rangle + \left(|0\rangle + |1\rangle\right)|1\rangle\right) =$

$\frac{1}{2}\left(|01\rangle - |11\rangle + |01\rangle + |11\rangle\right) = |01\rangle$

Bob now measures both qubits to get Alice's message 01

$|\psi_{10}\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle - |11\rangle\right) = \frac{1}{\sqrt{2}}\left(|0\rangle|0\rangle - |1\rangle|1\rangle\right)$

apply $CNot$ gives us: $\frac{1}{\sqrt{2}}\left(|0\rangle|0\rangle - |1\rangle|0\rangle\right)$

applying H on 1st qubit gives us: $\frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}\left(\left(|0\rangle + |1\rangle\right)|0\rangle + \left(|1\rangle - |0\rangle\right)|0\rangle\right) =$

$\frac{1}{2}\left(|00\rangle + |10\rangle + |10\rangle - |00\rangle\right) = |10\rangle$

Bob now measures both qubits to get Alice's message 10

Check the last one, $|\psi_{11}\rangle \longrightarrow 11$, yourself.

3 No Cloning Theorem

A quantum operation which copied states would be very useful. For example, we considered the following problem in Homework 1: Given an unknown quantum state, either $|\phi\rangle$ or $|\psi\rangle$, use a measurement to guess which one. If $|\phi\rangle$ and $|\psi\rangle$ are not orthogonal, then no measurement perfectly distinguishes them, and we always have some constant probability of error. However, if we could make many copies of the unknown state, then we could repeat the optimal measurement many times, and make the probability of error arbitrarily small. The no cloning theorem says that this isn't physically possible. Only sets of mutually orthogonal states can be copied by a single unitary operator.

There are two ways to prove the no cloning theorem. The first follows from the norm preserving property of the inner product, the second from the linearity of quantum mechanics.

No Cloning Assume we have a unitary operator U_{cl} and two quantum states $|\phi\rangle$ and $|\psi\rangle$ which U_{cl} copies, i.e.,

$$\begin{aligned} |\phi\rangle \otimes |0\rangle &\xrightarrow{U_{cl}} |\phi\rangle \otimes |\phi\rangle \\ |\psi\rangle \otimes |0\rangle &\xrightarrow{U_{cl}} |\psi\rangle \otimes |\psi\rangle \ . \end{aligned}$$

Then $\langle\phi|\psi\rangle$ is 0 or 1.

Proof 1: $\langle\phi|\psi\rangle = (\langle\phi|\otimes\langle 0|)(|\psi\rangle\otimes|0\rangle) = (\langle\phi|\otimes\langle\phi|)(|\psi\rangle\otimes|\psi\rangle) = \langle\phi|\psi\rangle^2$. In the second equality we used the fact that U, being unitary, preserves inner products. □

Proof 2: Suppose there exists a unitary operator U_{cl} that can indeed clone an unknown quantum state $|\phi\rangle = \alpha|0\rangle + \beta|1\rangle$. Then

$$\begin{aligned} |\phi\rangle|0\rangle \xrightarrow{Ucl} |\phi\rangle|\phi\rangle &= (\alpha|0\rangle + \beta|1\rangle)(\alpha|0\rangle + \beta|1\rangle) \\ &= \alpha^2|00\rangle + \beta\alpha|10\rangle + \alpha\beta|01\rangle + \beta^2|11\rangle \end{aligned}$$

But now if we use U_{cl} to clone the expansion of $|\phi\rangle$, we arrive at a different state:

$$(\alpha|0\rangle + \beta|1\rangle)|0\rangle \xrightarrow{U_{cl}} \alpha|00\rangle + \beta|11\rangle .$$

Here there are no cross terms. Thus we have a contradiction and therefore there cannot exist such a unitary operator U_{cl}. □

Note that it is however possible to clone a known state such as $|0\rangle$ and $|1\rangle$.

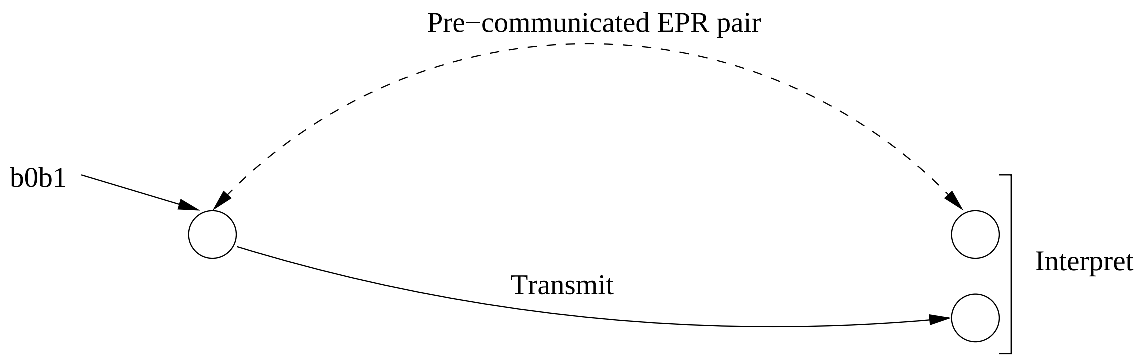

Figure 1: Superdense coding works by first pre-communicating an EPR pair between sender (Alice, on left) and receiver (Bob, on right). Alice first manipulates her half of this EPR pair (a single qubit) according to which two classical bit message she wishes to transmit, and then sends her qubit to Bob. Bob then makes some two qubit manipulations and finally measures both bits to obtain the two bits of classical information.

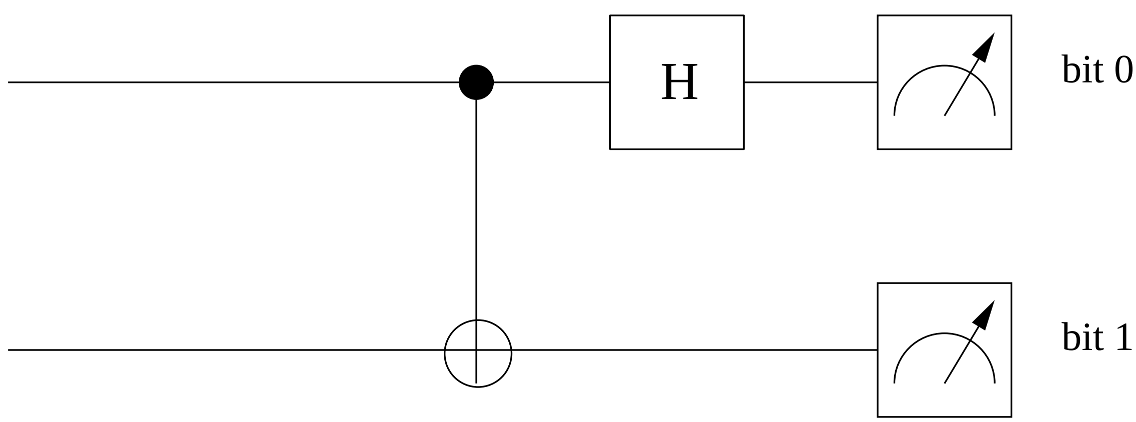

Figure 2: To obtain the two bits of classical information this circuit is implemented by Bob on the two qubits, one of which has been sent by Alice. Bob's circuit is equivalent to making a measurement in the Bell-basis.