Quantum Simulation

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Quantum Simulation

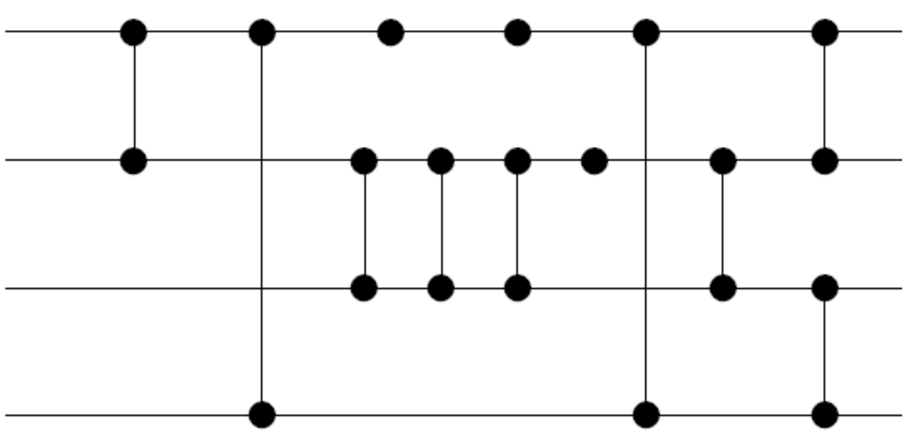
- Studies the efficient classical simulation of quantum circuits
- Understanding what is (and is not) efficiently simulable has implications to the usefulness of quantum circuits
- Known to be closely related to entanglement
- In general, simulation is exponential but some classes of circuits may be done efficiently

Three Results in Quantum Simulation

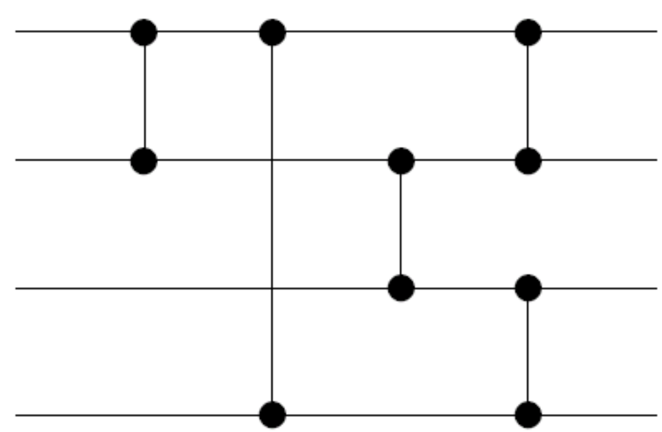
- Any poly sized (and generally poly depth) circuit on n-qubits can be efficiently simulated provided the maximum number of gates touching or spanning any wire is bounded by O(log n). (Jozsa, 2006)
- The evolution of a quantum state can be simulated with resources linear in the size of the state and exponential in the the entanglement. If the maximum Schmidt-rank of the system is poly in the size of the state, then the simulation is efficient. (Vidal, 2003)
- Multi-scale Entanglement Renormalization Ansatz (MERA) is a way of describing a quantum circuit as a lattice with special properties that make it classically simulable (Vidal, 2006)

Quantum Circuit Reduction

- Reduce quantum circuits to having only 1- and 2-qubit gates
- Formulate as parallel wires, and reduce it by merging all 1-qubit gates into neighboring gates, and merging all consecutive 2-qubit gates that can be performed sequentially without an interposing gate.



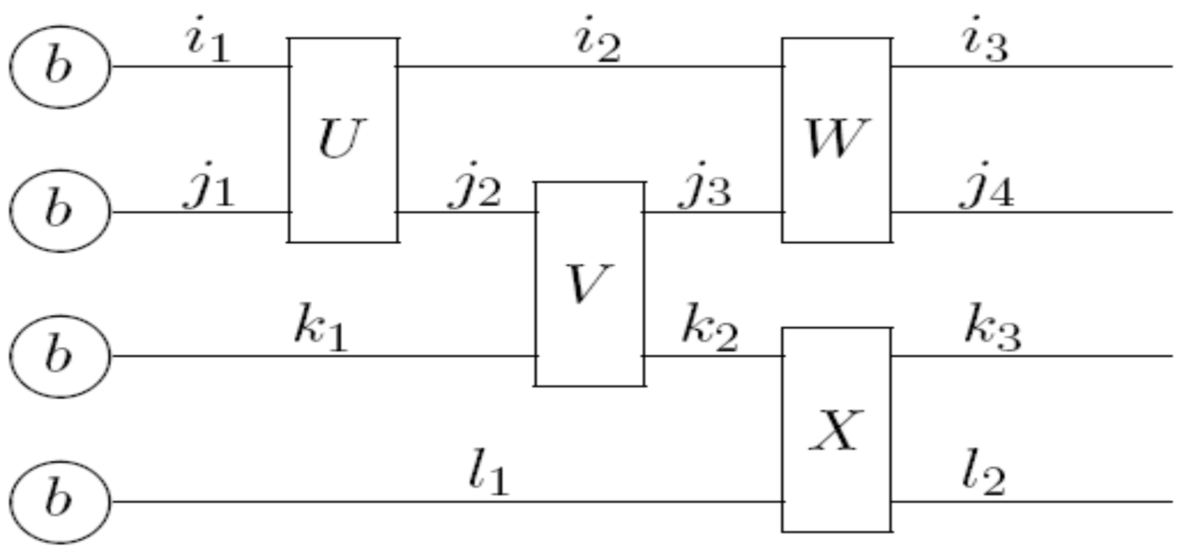
A circuit $\mathcal{C}$



The reduced form $\mathcal{C}_{red}$

Contracting Linear Networks

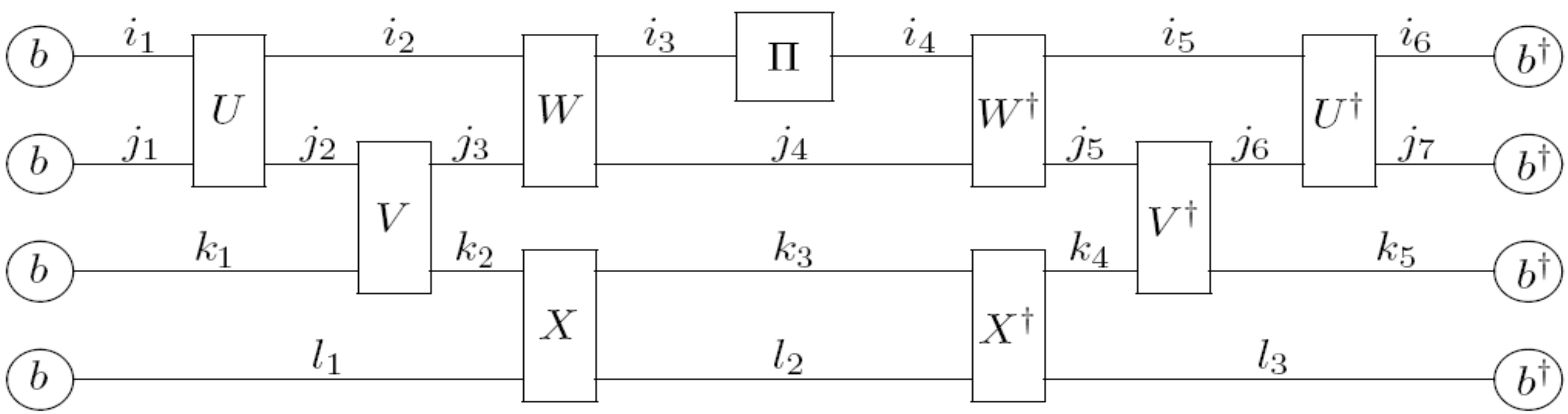
- For each wire, let Di be the number of 2-qubit gates that touch or cross it. Then, D = maxi Di.



$$A^{i_3 j_4 k_3 l_2} = b^{i_1} b^{j_1} b^{k_1} b^{l_1} U^{i_2 j_2}_{i_1 j_1} V^{j_3 k_2}_{j_2 k_1} W^{i_3 j_4}_{i_2 j_3} X^{l_2 k_3}_{l_1 k_2}$$

Contracting Linear Networks (efficient computation)

- We measure with the projector matrix $\Pi_n^m = |k\rangle\langle k|$



- The output is prob(k) can be easily computed from the full transformation.
- This sum takes time O(2^D), so doing so for all wires takes time O(n 2^D) which is polynomial in n if D = O(log n).

Schmidt Decomposition

- As a consequence from linear algebra (related to SVD) we can decompose any quantum state $|\Psi\rangle \in \mathcal{H}_2^{\otimes n}$ into two subspaces $\mathcal{H}_2^{\otimes n} = \mathcal{H}_A \otimes \mathcal{H}_B$ where:

$$|\psi\rangle_{AB} = \sum_{i=1}^{n} \tilde{\chi_i} |u_i\rangle_A \otimes |v_i\rangle_B$$

- The number of non-zero Schmidt coefficients ($\tilde{\chi_i}$) is the Schmidt rank of the decomposition.
- The state is separable iff it has a Schmidt rank of one.
- If more than one coefficient is non-zero, the state is entangled. If they are all non-zero, it is maximally entangled.
- Thus the maximum Schmidt rank makes a good measure of the state's entanglement.

Slightly entangled states

"We say a pure-state quantum evolution is *slightly entangled* if, at all times t, the state $|\Psi(t)\rangle$ of the system is slightly entangled – that is, if $\chi(t)$ is small. A sequence of evolutions with an increasingly large number of *n* qubits is *slightly entangled* if $\chi_n(t)$ is upper bounded by poly(*n*)." (Vidal, 2003)

- By restricting ourselves to circuits with limited entanglement, we can prove them to be efficiently simulable.
- Conversely, we can place a lower bound on the entanglement in useful quantum circuits.

Local state decomposition

- In general, representing $|\Psi\rangle \in \mathcal{H}_2^{\otimes n}$ requires exponential state. If it is only slightly entangled, we can inductively use Schmidt decomposition to get:

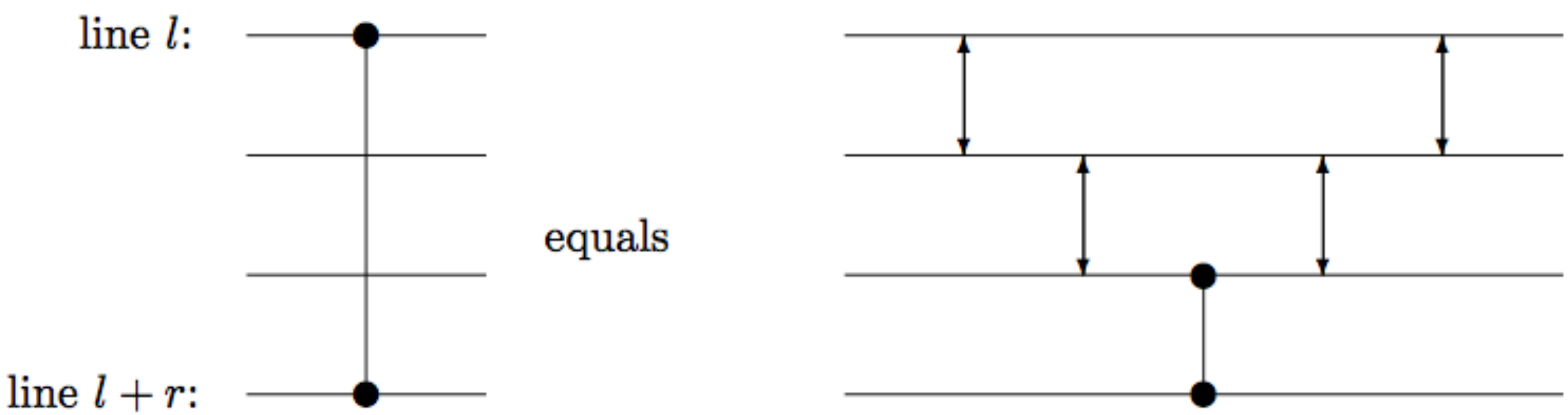
$$|\Psi\rangle \longleftrightarrow \Gamma^{[1]}\lambda^{[1]}\Gamma^{[2]}\lambda^{[2]}\ldots\Gamma^{[l]}\ldots\lambda^{[n-1]}\Gamma^{[n]}$$

- Which is specified in terms of $(2\chi^2 + \chi)n$ parameters.
- This representation requires we pick an ordering of qubits, but once we've done so we can keep updates local. That is to apply a gate to qubits *l* and *l+1* we need only update

$$\Gamma^{[l]},\ \lambda^{[l]},\ \text{and}\ \Gamma^{[l+1]}$$

Mind the gap

- Problem: Our decomposed representation allows for efficient local update, but gates can span multiple wires.
- Solution: decompose a gate spanning r wires into O(r) gates spanning two adjacent wires.

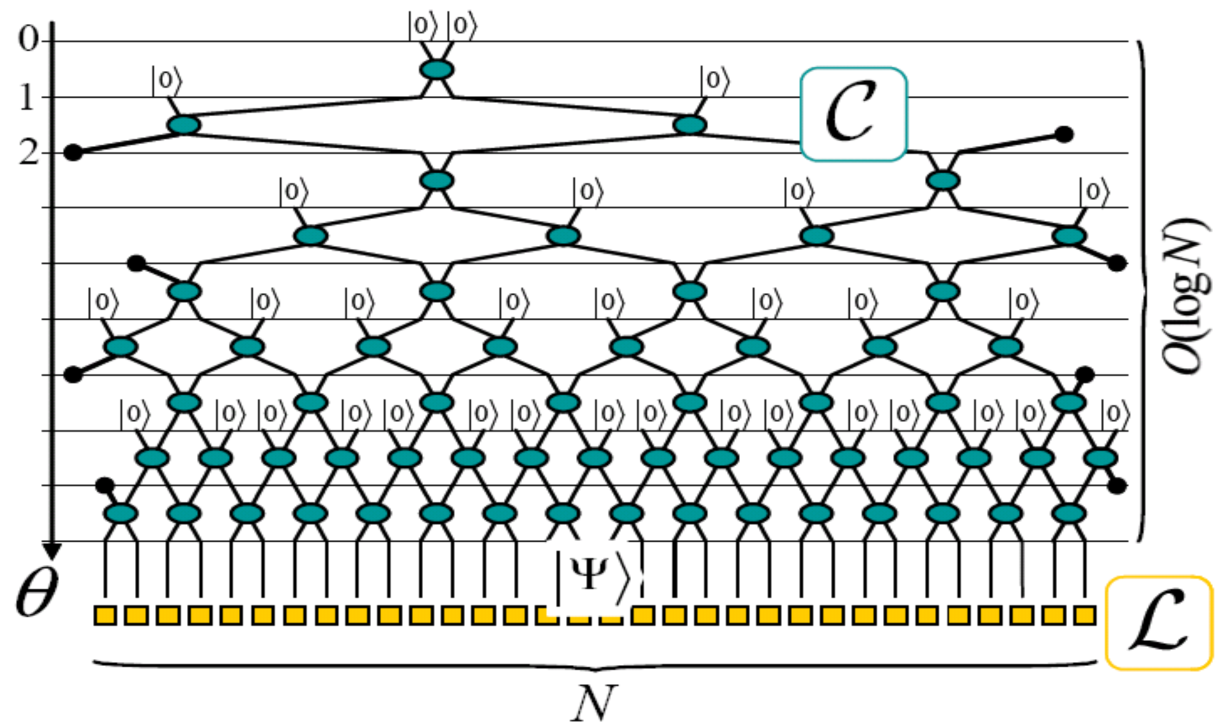


Efficient updates

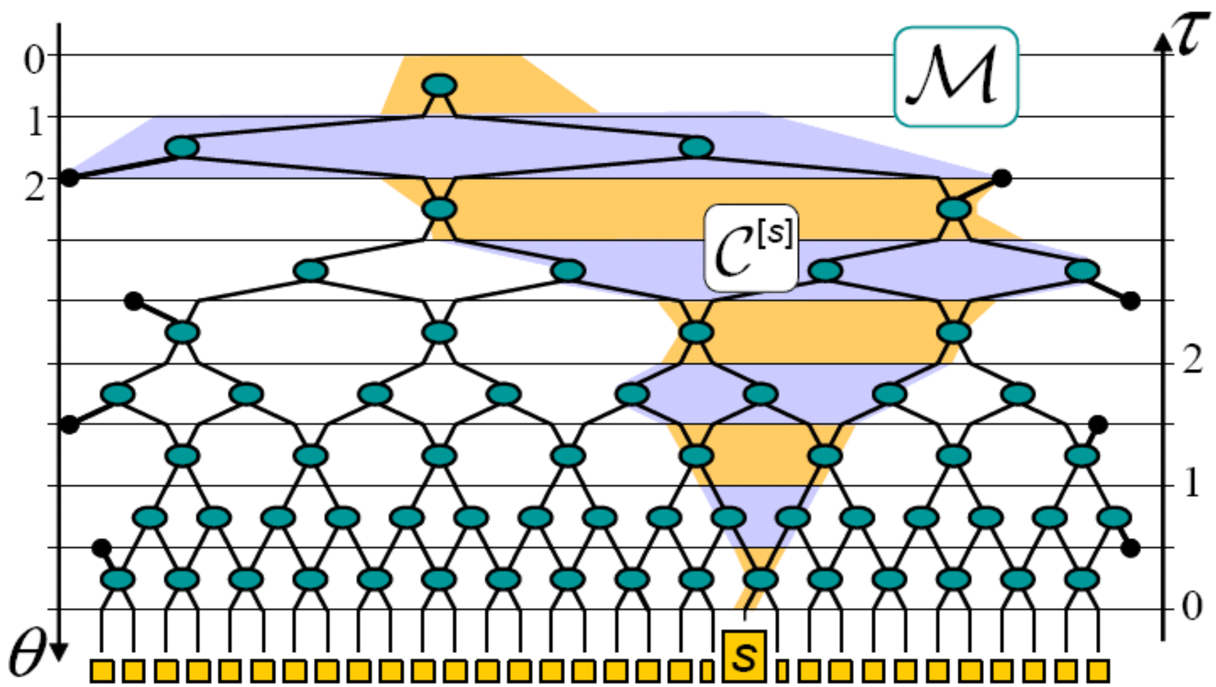
- Single qubit gates only transform $\Gamma^{[l]}$ and can be computed in $O(\chi^2)$ time.
- Proof sketch: consider that a unitary operation U doesn't affect the Schmidt decomposition above or below *l* .
- By a similar (but more involved) argument, it is possible to show that applying a gate to *l* and *l+1* can be done in $O(\chi^3)$ time and $O(\chi^2)$ space.
- Thus, provided χ is constrained to grow slowly, the circuit can be efficiently simulated.
- Note: global entanglement appears key to computational speed up.

Multi-scale Entanglement Renormalization Ansatz (MERA)

- A MERA efficiently encodes a lattice of quantum states of spatial dimension, D.
- The MERA for $|\Psi\rangle$ is a tensor network, M, that corresponds to quantum circuit, C.
- The dimension of the q-states in $|\Psi\rangle$ is X.



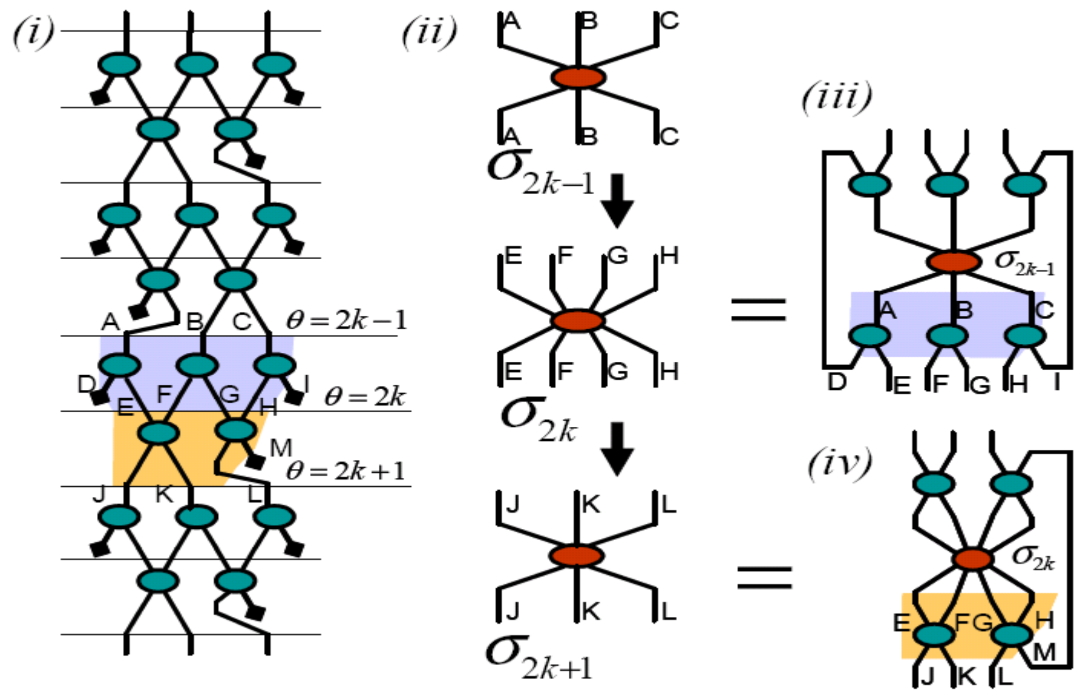
MERAs: Causal Cone

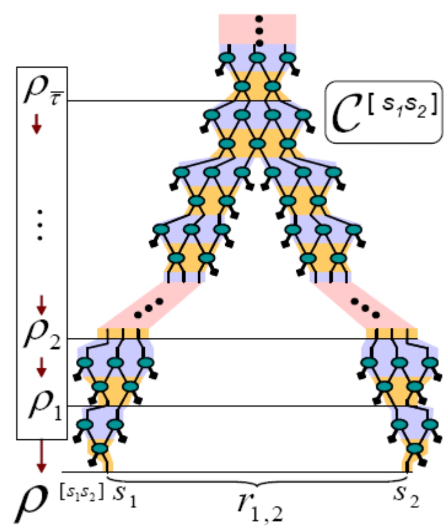


- C[s] includes the gates and wires that influence p[s]
- Ctheta[s], the number of wires in a time slice of C[s], is bounded by 3^(D-1)*4

MERAs: Computing p[s] and p[s1s2]

- The density matrix for each slice of Ctheta[s] and Ctheta [s1s2] can be computed from the density matrix for the previous slice in time polynomial to X.
- Since there are logN slices, p[s] and p [s1s2] (generally: p [s1...sn]) can be computed in time O (X*logN).





Conclusions

- Quantum simulation is (necessarily) hard
- Understanding what can be efficiently simulated places lower bounds on worthwhile quantum circuits
- Entanglement appears to be the limiting factor in simulation
- Better models constraining entanglement enable efficient algorithms; the mathematical models can be thought of as efficient data-structures.

Questions?